Symmetry classes for odd-order tensors

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We give a complete general answer to the problem, recurrent in continuum mechanics, of determining the number and type of symmetry classes of an odd-order tensor space. This kind of investigation was initiated for the space of elasticity tensors. Since then, this problem has been solved for other kinds of physics such as photoelectricity, piezoelectricity, flexoelectricity, and strain-gradient elasticity. In all the aforementioned papers, the results are obtained after some lengthy computations. In a former contribution we provide general theorems that solve the problem for even-order tensor spaces. In this paper we extend these results to the situation of odd-order tensor spaces. As an illustration of this method, and for the first time, the symmetry classes of all odd-order tensors of Mindlin second strain-gradient elasticity are provided.

1 Introduction

1.1 Physical motivation

In the last years there has been increased interest in generalized continuum theories; see, for example, [10–12, 24]. These works, based on the pioneering articles [26, 27, 32], propose an extended kinematic formulation to take into account size effects within the continuum. The price to be paid for this is the appearance of tensors of order greater than four in the constitutive relations. These higher-order objects are difficult to handle, and extracting physical meaningful information information from them is not straightforward. In a previous paper [28] we provide general theorems concerning the symmetry classes determination of even-order tensor spaces. The aim of this paper is to extend this approach and to provide a general theorem concerning the type and number of anisotropic systems an odd order-tensor space can have.

Constitutive tensors symmetry classes

In mechanics, constitutive laws are usually expressed in terms of tensorial relations between the gradients of primary variables and their fluxes [18]. As is well-known, this feature is not restricted to linear behaviors since tensorial relations appear in the tangential formulation of non-linear ones [33]. It is also known that a general tensorial relation can be divided into classes according to its symmetry properties. Such classes are known in mechanics as symmetry classes [14], and in mathematical physics as isotropic classes or strata [1, 4].

In the case of second order tensors, the determination of symmetry classes is rather simple. Using spectral analysis it can be concluded that any second-order symmetric tensor[1] can either be orthotropic ($[\mathrm{D}_2]$), transverse isotropic ($[\mathrm{O}(2)]$), or isotropic ($[\mathrm{SO}(3)]$). Such tensors are commonly used to describe, for example, the heat conduction, the electric permittivity, etc. For higher-order tensors, the determination of the set of symmetry classes is more involved, and is mostly based on an approach introduced by Forte and Vianello [14] in the case of elasticity. Let us briefly detail this case.

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[1] Such a tensor is related to a symmetric matrix, which can be diagonalized in an orthogonal basis. The stated result is related to this diagonalization.

The vector space of elasticity tensors, denoted by $\mathbb{E}\text{la}$ throughout this paper, is the subspace of fourth-order tensors endowed with the following index symmetries:

Minor symmetries: : $E_{ijkl} = E_{jikl} = E_{jilk}$

Major symmetry: : $E_{ijkl} = E_{klij}$

Symmetries will be specified using notation such as: $E_{\underline{(ij)}\ \underline{(kl)}}$, where (..) indicates invariance under permutation of the indices in parentheses, and $\underline{..}\ \underline{..}$ indicates invariance with respect to permutations of the underlined blocks. Index symmetries encode the physics described by the tensor. The minor symmetries stem from the fact that rigid body motions do not induce deformation, and that the material is not subjected to volumic couple. The major symmetry is the consequence of the existence of a free energy. An elasticity tensor, $\mathbf{E}$, can be viewed as a symmetric linear operator on $\mathbb{T}_{(ij)}$, the space of symmetric second order tensors. According to Forte and Vianello [14], for the classical action of SO(3),

$$\forall(\mathrm{Q}, \mathbf{E}) \in \mathrm{SO}(3) \times \mathbb{E}\text{la}, \quad (\mathrm{Q} \star \mathbf{E})_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}E_{mnop}$$

$\mathbb{E}\text{la}$ is divided into the following eight symmetry classes:

$$\mathfrak{I}(\mathbb{E}\text{la}) = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{D}_3], [\mathrm{D}_4], [\mathrm{O}(2)], [\mathcal{O}], [\mathrm{SO}(3)]\}$$

which correspond, respectively, to the following physical classes[2]: triclinic ($[\mathbb{1}]$), monoclinic ($[\mathrm{Z}_2]$), orthotropic ($[\mathrm{D}_2]$), trigonal ($[\mathrm{D}_3]$), tetragonal ($[\mathrm{D}_4]$), transverse isotropic ($[\mathrm{O}(2)]$), cubic ($[\mathcal{O}]$) and isotropic ($[\mathrm{SO}(3)]$). The mathematical notations used above are detailed in subsection 2.4. Besides this fundamental result, the interest of the Forte and Vianello paper was to provide a general method to determine the symmetry classes of any tensor space [4]. Other results have been obtained by this method since then:

Property	Tensor	Number of classes	Action	Studied in
Photoelelasticity	$T_{(ij)(kl)}$	12	SO(3)	[15]
Piezoelectricity	$T_{(ij)k}$	15	O(3)	[16]
Flexoelectricity	$T_{(ij)kl}$	12	SO(3)	[23]
A set of 6-th order tensors	T_{ijklmn}	14 or 17	SO(3)	[22]

The main drawback of the original approach is the specificity of the study for each kind of tensor, and therefore no general results can be obtained. In a recent paper [28], we demonstrated a general theorem that solve the question for even-order tensor spaces. The aim of the present paper is to extend the former result to odd-order tensor spaces, and therefore to completely solve the question. From a mathematical point of view, the analysis will differ from [28] as the group action is O(3) instead of SO(3).

1.2 Organization of the paper

In section 2, the main results of this paper are stated. As an application, the symmetry classes of the odd-order constitutive tensor spaces of Mindlin second strain-gradient elasticity are determined [27]. Results concerning the 5th-order and the 7th-order coupling tensors are given for the first time. Obtaining the same results with the Forte-Vianello approach would have been much more difficult. Other sections are dedicated to the construction of our proofs. In section 3, the mathematical framework used to obtain our result is introduced. The main purpose of section 4 is to introduce a tool named *clips operator*, which constitutes the cornerstone of our demonstration. Definition of the clips operation and the associated results for couples of SO(3)- and O(3)-closed subgroups are provided in this section. Sketches of proofs of these results and calculus of clips operations are postponed to the Appendix[3]. Using these results in section 5 the theorem stated in section 2 is proved.

2 Main results

In this section, our main results are stated. In the first subsection, the construction of constitutive tensor spaces (CTS) is discussed. This construction allows us to formulate our main results in the next subsection. Finally, the application of these results to Mindlin second strain-gradient elasticity (SSGE) is considered. Precise mathematical definitions of the symmetry classes are given in Section 3.

[2] These symmetry classes are subgroups of the group SO(3) of space rotations. This is because the elasticity tensor is of even order. To treat odd-order tensors, the full orthogonal group O(3) has to be considered.

[3] For more details on the main ideas of the proofs related to clips operations, the reader is referred to [28].

2.1 Construction of CRS

Linear constitutive laws are linear maps between the gradients of primary physical quantities and their fluxes [36]. Each of these physical quantities is related to subspaces[4] of tensor spaces; theses subspaces will be called *state tensor spaces* (STS in the following). These STS will be the primitive notion from which the CTS will be constructed. In the following, $\mathcal{L}(F, G)$ will indicate the vector space of linear maps from F to G.

Physical notions	Mathematical object	Mathematical space
Gradient	Tensor state $T_1 \in \otimes^p \mathbb{R}^3$	$\mathbb{T}_G$: tensor space with index symmetries
Fluxes of gradient	Tensor state $T_2 \in \otimes^q \mathbb{R}^3$	$\mathbb{T}_f$: tensor space with index symmetries
Linear constitutive law	$C \in \mathcal{L}(\mathbb{T}_G, \mathbb{T}_f) \sim \mathbb{T}_G \otimes \mathbb{T}_f$	$\mathbb{T}_c \subset \mathbb{T}_G \otimes \mathbb{T}_f$

Now we consider two STS: $\mathbb{E}_1 = \mathbb{T}_G$ and $\mathbb{E}_2 = \mathbb{T}_f$, respectively of order p and order q, possibly with index symmetries. As a consequence, they belong to subspaces of $\otimes^p \mathbb{R}^3$ and $\otimes^q \mathbb{R}^3$, where the notation $\otimes^k \mathbb{R}^n$ indicates that the space is generated by k tensor products of $\mathbb{R}^n$. *A constitutive tensor* C is a linear map between $\mathbb{E}_1$ and $\mathbb{E}_2$, that is, an element of the space $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$. This space is isomorphic, modulo the use of an euclidean metric, to $\mathbb{E}_1 \otimes \mathbb{E}_2$. Physical properties lead to some index symmetries on $C \in \mathbb{E}_1 \otimes \mathbb{E}_2$; thus the vector space of such C is some vector subspace $\mathbb{T}_c$ of $\mathbb{E}_1 \otimes \mathbb{E}_2$. Now, each of the spaces $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{E}_1 \otimes \mathbb{E}_2$ has a natural $\mathrm{O}(3)$-action. We therefore have

$$\mathcal{L}(\mathbb{E}_1, \mathbb{E}_2) \simeq \mathbb{E}_1 \otimes \mathbb{E}_2 \subset \mathbb{T}^p \otimes \mathbb{T}^q = \mathbb{T}_c^{p+q}$$

Here are some examples of this construction:

Property	$\mathbb{E}_1$	$\mathbb{E}_2$	Tensor product for CTS	Number of classes
Elasticity	$\mathbb{T}_{(ij)}$	$\mathbb{T}_{(ij)}$	Symmetric: $\otimes^s$	8
Photoelelasticity	$\mathbb{T}_{(ij)}$	$\mathbb{T}_{(ij)}$	Standard: $\otimes$	12
Flexoelectricity	$\mathbb{T}_{(ij)k}$	$\mathbb{T}_i$	Standard: $\otimes$	12
First-gradient elasticity	$\mathbb{T}_{(ij)k}$	$\mathbb{T}_{(ij)k}$	Symmetric: $\otimes^s$	17

This table shows two kinds of CTS, describing respectively

- Coupled physics: tensors such as photoelasticity and flexoelectricity, encoding the coupling between two different physics;
- Proper physics: tensors such as classical and first-gradient elasticities, describing a single physical phenomenon.

On the mathematical side this implies:

- Coupled physics: the spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ may differ, and when $\mathbb{E}_1 = \mathbb{E}_2$ linear maps are not self-adjoint, therefore the tensor product is standard: $\otimes$;
- Proper physics: we have $\mathbb{E}_1 = \mathbb{E}_2$ and linear maps are self-adjoint[5], in which case the tensor product is symmetric: $\otimes^s$;

Therefore, the *elasticity tensor* is a self-adjoint linear map between the vector spaces of deformation and stress tensors. These two spaces are modeled on $\mathbb{T}_{(ij)}$. The vector space of elasticity tensors is therefore completely determined by $\mathbb{T}_{(ij)}$ and the symmetric nature of the tensor product, that is, $\mathbb{E}\mathrm{la} = \mathbb{T}_{(ij)} \otimes^s \mathbb{T}_{(kl)}$. In this paper, we are only concerned with cases in which $p + q = 2n + 1$. Obviously, odd-order tensors can only describe coupling tensors. On a mathematical side this implies that the spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ are of different parity and that the tensor space $\mathbb{T}_c$ is $\mathbb{E}_1 \otimes \mathbb{E}_2$. For example, the piezoelectricity is a linear map between $\mathbb{E}_1 = \mathbb{T}_{(ij)}$, the space of deformation and $\mathbb{E}_2 = \mathbb{T}_k$ the electric polarization, therefore $\mathbb{P}\mathrm{iez} = \mathbb{T}_{(ij)} \otimes \mathbb{T}_k$ [34].

To properly present our main result, the next two subsections are devoted to the introduction of some technical concepts needed to state our theorem. In the next subsection the concept of harmonic decomposition is presented together with an effective method to easily compute its structure. As the symmetry group of an odd-order tensor is conjugate to a $\mathrm{O}(3)$-closed subgroup [37], the subsection subsection 2.3 is dedicated to their presentation.

[4] Because of some symmetries.

[5] This is a consequence of the assumption of the existence of a free energy.

2.2 The harmonic decomposition

To proceed our analysis, we need to break the tensor spaces we study into elementary building blocks which mathematically correspond to irreducible subspaces. The definition of these elementary spaces depends on the considered group action. In the present situation, tensor spaces will be decomposed into O(3)-invariant spaces. Such a decomposition has been widely used in the mechanical community for studying anisotropic elasticity [5, 6, 14, 15] and is often referred to as the harmonic decomposition. In order to present our results in self-contained way, basic definitions of the harmonic decomposition are summed up here. A more general and rigorous[6] presentation of this object can be found in subsection 3.3.

In $\mathbb{R}^3$, under O(3)-action, any tensor space V can be decomposed orthogonally into a sum of harmonic tensor spaces of different orders[7] of V:

$$V = \bigoplus_{i=0}^{p} \alpha_i \mathbb{H}^i$$

where p indicates the tensorial order of V, $\mathbb{H}^i$ is the vector space of i-th order harmonic tensors and α_i indicates the number of copies of $\mathbb{H}^i$ in the decomposition. Elements of $\mathbb{H}^i$ are i-th order completely symmetric traceless tensors, the dimension of their vector space is $2i+1$. The denomination harmonic is related to a classical isomorphism[8] between the space of harmonic polynomials of degree i and the space of i-th order completely symmetric traceless tensors, all considered in $\mathbb{R}^3$. A nice and comprehensive presentation of this construction can be found in [14].

For example, it can be shown, as demonstrated by Forte and Vianello [14], that:

$$\mathbb{E}\mathrm{la} = 2\mathbb{H}^0 + 2\mathbb{H}^2 + \mathbb{H}^4$$

furthermore, in this publication, using an algorithm formerly introduced by Spencer [29] they explicitly construct the isomorphism. It worths being noted that such an isomorphism is uniquely defined iff $\alpha_i \leq 1$ for $i \in [0, p]$. According to the problem under investigation, the explicit knowledge of an isomorphism that realize the decomposition may be required or not. This point is really important because the explicit computation of an isomorphism is a task which complexity increases very quickly with the tensorial order. At the opposite, the determination of structure of the decomposition, i.e. the number of spaces and their multiplicities is almost straightforward using the Clebsch-Gordan product.

For our need, and in order to establish the symmetry class decomposition of a tensor space, we only need to known the structure of the harmonic decomposition. Therefore we can bypass the heavy computational step of the initial Forte and Vianello procedure. Thus, to compute the harmonic structure of a tensor space we use the tensorial product of group representations. More details can be found in [2, 21]. The computation rule is simple. Let's consider two harmonic tensor spaces $\mathbb{H}^i$ and $\mathbb{H}^j$, whose product space is noted $\mathbb{T}^{i+j} := \mathbb{H}^i \otimes \mathbb{H}^j$. This space, which is GL(3)-invariant, admits the following O(3)-invariant decomposition:

$$\mathbb{T}^{i+j} = \mathbb{H}^i \otimes \mathbb{H}^j \simeq \bigoplus_{k=|i-j|}^{i+j} \mathbb{H}^k \tag{1}$$

In the following of this paper, this rule will be referred to as the Clebsh-Gordan product.

For example, consider $\mathbb{H}^1_a$ and $\mathbb{H}^1_b$ two different first-order harmonic spaces. Elements of such spaces are vectors. According to formula (1), the O(3)-invariant decomposition of $\mathbb{T}^2$ is:

$$\mathbb{T}^2 = \mathbb{H}^1_a \otimes \mathbb{H}^1_b = \mathbb{H}^2 \oplus \mathbb{H}^{\sharp 1} \oplus \mathbb{H}^0$$

i.e. the tensorial product of two vector spaces generates a second-order tensor space which decompose as scalar ($\mathbb{H}^0$), a vector ($\mathbb{H}^{\sharp 1}$) and a deviator ($\mathbb{H}^2$). Spaces indicated with $\sharp$ contain pseudo-tensors,

[6] In order to introduce the harmonic decomposition as an operative tool, we do not mention in this subsection the 2 different representations of O(3) on a vector space. Nevertheless, and in order to be rigorous, this point is discussed in subsection 3.3.

[7] In the harmonic decomposition of a tensor space, the equality sign means that the exists an isomorphism between the right- and the left-hand side of the decomposition. In order to avoid the use of to many notation we do not use a specific sign to indicate this isomorphism.

[8] In [5] Backus investigates this isomorphism and shows that this association is no more valid in $\mathbb{R}^{N\geq 4}$.

also known as axial-tensors that is tensors which change sign if the space orientation is reversed. Other elements are true tensors, also known as polar, and transform according the usual rules.

This computation rule has to be completed by the following properties [21]

Property 2.1 *The decomposition of an even-order (resp. odd-order) completely symmetric tensor only contains even-order (resp. odd-order) harmonic spaces.*

Property 2.2 *In the decomposition of an even-order (resp. odd-order) tensor, even-order components are polar (resp. axial) and odd-order component are axial (resp. polar).*

2.3 O(3)-closed subgroups

The symmetry group of an odd-order tensor is conjugate to a O(3)-closed subgroup [14,37]. Classification of O(3)-closed subgroups is a classical result that can be found in many references [20,30]:

Lemma 2.3 *Every closed subgroup of* $\mathrm{O}(3)$ *is conjugate to precisely one group of the following list, which has been divided into three classes:*

(I) Closed subgroups of $\mathrm{SO}(3)$*:* $\{\mathbb{1}\}$, Z_n, D_n, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$, $\mathrm{SO}(2)$, $\mathrm{O}(2)$, $\mathrm{SO}(3)$*;*

(II) $\tilde{K} := K \oplus \mathrm{Z}_2^c$*, where* K *is a closed subgroup of* $\mathrm{SO}(3)$ *and* $\mathrm{Z}_2^c = \{\mathbb{1}, -\mathbb{1}\}$*;*

(III) Closed subgroups not containing $-\mathbb{1}$ *and not contained in* $\mathrm{SO}(3)$*:*

$$\mathrm{Z}_{2n}^- \ (n \geq 1),\ \mathrm{D}_n^v \ (n \geq 2),\ \mathrm{D}_{2n}^h \ (n \geq 2),\ \mathcal{O}^- \text{ or } \mathrm{O}(2)^-$$

Type I subgroups

Among SO(3)-closed subgroups we can distinguish:

Planar groups: : $\{\mathbb{1}, \mathrm{Z}_n, \mathrm{D}_n, \mathrm{SO}(2), \mathrm{O}(2)\}$, which are O(2)-closed subgroups;

Exceptional groups: : $\{\mathcal{T}, \mathcal{O}, \mathcal{I}, \mathrm{SO}(3)\}$, which are symmetry groups of chiral Platonic polyhedrons completed by the rotation group of the sphere.

Let us detail first the set of planar subgroups. We fix a base $(\mathbf{i};\mathbf{j};\mathbf{k})$ of $\mathbb{R}^3$, and denote by $\mathbf{Q}(\mathbf{v};\theta) \in \mathrm{SO}(3)$ the rotation about $\mathbf{v} \in \mathbb{R}^3$, with angle $\theta \in [0; 2\pi)$ we have

- $\mathbb{1}$, the identity;
- Z_n $(n \geq 2)$, the cyclic group of order n, generated by the n-fold rotation $\mathbf{Q}\left(\mathbf{k}; \theta = \dfrac{2\pi}{n}\right)$, which is the symmetry group of a chiral polygon;
- D_n $(n \geq 2)$, the dihedral group of order $2n$ generated by Z_n and $\mathbf{Q}(\mathbf{i};\pi)$, which is the symmetry group of a regular polygon;
- $\mathrm{SO}(2)$, the subgroup of rotations $\mathbf{Q}(\mathbf{k};\theta)$ with $\theta \in [0; 2\pi)$;
- $\mathrm{O}(2)$, the subgroup generated by $\mathrm{SO}(2)$ and $\mathbf{Q}(\mathbf{i};\pi)$.

The classes of exceptional subgroups are: $\mathcal{T}$ the tetrahedral group of order 12 which fixes a tetrahedron, $\mathcal{O}$ the octahedral group of order 24 which fixes an octahedron (or a cube), and $\mathcal{I}$ the subgroup of order 60 which fixes an icosahedron (or a dodecahedron).

Type II subgroups

Type II subgroups are of the form $\tilde{K} := K \oplus \mathrm{Z}_2^c$, where K is a closed subgroup of $\mathrm{SO}(3)$. Therefore we directly know the collection of type II subgroups.

Type III subgroups

The construction of type III ones is more involved, and a short description of their structure is provided in Appendix A. As for type I subgroups, we can introduce subgroups of type III. Let $\sigma_{\mathbf{u}} \in \mathrm{O}(3)$ denotes the reflection through the plane normal to $\mathbf{u}$ axis. Then:

- Z_2^- is the order 2 reflection group generated by $\sigma_{\mathbf{i}}$;
- Z_{2n}^- $(n \geq 2)$ is the group of order $2n$, generated by the $2n$-fold rotoreflection $\mathbf{Q}\left(\mathbf{k};\theta = \dfrac{\pi}{n}\right) \cdot \sigma_{\mathbf{k}}$;
- D_{2n}^h $(n \geq 2)$ is the prismatic group of order $4n$ generated by Z_{2n}^- and $\mathbf{Q}(\mathbf{i},\pi)$. When n is odd it is the symmetry group of a regular prism, and when n is even it is the symmetry group of a regular antiprism;
- D_n^v $(n \geq 2)$ is the pyramidal group of order $2n$ generated by Z_n and $\sigma_{\mathbf{i}}$, which is the symmetry group of a regular pyramid;
- $\mathrm{O}(2)^-$ is the limit group of D_n^v for continuous relation, it is therefore generated by $\mathbf{Q}(\mathbf{k};\theta)$ and $\sigma_{\mathbf{i}}$. It is the symmetry group of a cone;

These planar subgroups are completed by the achiral tetrahedral symmetry $\mathcal{O}^-$ which is of order 24. This group has the same rotation axes as $\mathcal{T}$, but with six mirror planes, each through two 3-fold axes. In order to have a better understanding of these subgroups, in Appendix B tables making correspondences between group notations and the classical crystallographic ones (Hermann-Mauguin, Schoenflies) are provided.

Now we can go back to our initial problem concerning the determination of the symmetry classes of odd-order tensor spaces.

2.4 Symmetry classes of odd-order tensor spaces

Consider an odd-order constitutive tensor space $\mathbb{T}_c^{2n+1}$, i.e. a tensor space obtained by the tensor product of two other spaces: $\mathbb{T}_c^{2n+1} = \mathbb{T}^{2p+1} \otimes \mathbb{T}^{2q}$.

Let us introduce:

$\mathbb{S}^{2n+1}$: the vector space of $2n+1$th-order completely symmetric tensors;

$\mathbb{G}^{2n+1}$: the vector space of $2n+1$th-order tensors with no index symmetries, that is $\mathbb{G}^k := \otimes^k \mathbb{R}^3$;

$\mathbb{G}_\star^{2n+1}$: the subspace of $\mathbb{G}^{2n+1}$ in which the scalar part have been removed[9]

The following inclusion will be demonstrated[10] in Section 5:

$$\mathbb{S}^{2n+1} \subset \mathbb{T}_c^{2n+1} \subseteq \mathbb{G}^{2n+1}$$

and therefore, if we denote by $\mathfrak{I}$ the operator which gives to a tensor space the set of its symmetry classes, we obtain:

$$\mathfrak{I}(\mathbb{S}^{2n+1}) \subset \mathfrak{I}(\mathbb{T}_c^{2n+1}) \subseteq \mathfrak{I}(\mathbb{G}^{2n+1})$$

For convenience, the collection of isotropy classes will be given considering apart spatial and planar classes. To that aim we define $\mathcal{P}$ and $\mathcal{S}$ to be the operators that extract from a set of symmetry classes, respectively, the collection of planar and spatial classes. In section 5, the symmetry classes of $\mathbb{G}^{2n+1}$ and $\mathbb{G}_\star^{2n+1}$ are obtained. We now provide in the following an outline of these results.

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[9] If we consider the harmonic decomposition of $\mathbb{G}^{2n+1}$ to be $\bigoplus_{i=0}^{p} \alpha_i \mathbb{H}^i$, the harmonic decomposition of $\mathbb{G}_\star^{2n+1}$ would start at $i=1$ instead of 0 and, therefore, $\bigoplus_{i=1}^{p} \alpha_i \mathbb{H}^i$. In other terms $\mathbb{G}^{2n+1} = \alpha_0 \mathbb{H}^0 \oplus \mathbb{G}_\star^{2n+1}$.

[10] The construction process of constitutive tensor spaces implies that no complete symmetric tensors space can be generated in this way.

2.4.1 Generic tensor space

For the generic tensor space:

Lemma 2.4 *The planar symmetry classes*[11] *of* $\mathbb{G}^{2n+1}$ *are:*

$$\begin{aligned}
\mathcal{P}(\mathfrak{I}(\mathbb{G}^1)) &= \{[\mathrm{O}(2)^-]\} \\
n \geq 1,\ \mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1})) &= \{[\mathbb{1}], [\mathrm{Z}_2], \cdots, [\mathrm{Z}_{2n+1}], [\mathrm{D}^v_2], \cdots, [\mathrm{D}^v_{2n+1}], [\mathrm{Z}^-_2], \cdots, [\mathrm{Z}^-_{4n}], [\mathrm{D}_2], \cdots, [\mathrm{D}_{2n+1}], \\
&\quad [\mathrm{D}^h_4], \cdots, [\mathrm{D}^h_{2(2n+1)}], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathrm{SO}(3)], [\mathrm{O}(3)]\}
\end{aligned}$$

with the following cardinality:

$$\forall n \geq 2,\ \#\mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1})) = 10n + 6$$

Lemma 2.5 *The spatial symmetry classes of* $\mathbb{G}^{2n+1}$ *are:*

n	1	2	≥ 3
$\mathcal{S}(\mathfrak{I}(\mathbb{G}^{2n+1}))$	$\{[\mathcal{O}^-]\}$	$\{[\mathcal{T}], [\mathcal{O}^-], [\mathcal{O}]\}$	$\{[\mathcal{T}], [\mathcal{O}^-], [\mathcal{O}], [\mathcal{I}]\}$

These results can be summed-up in the following table:

n	1	2	≥ 3
$\#\mathfrak{I}(\mathbb{G}^{2n+1})$	17	29	$10(n+1)$

2.4.2 Generic$_\star$ tensor space

$\mathbb{G}^{2n+1}_\star$ is the subspace of $\mathbb{G}^{2n+1}$ not containing scalars. The result is almost the same as in the previous case except that the class [SO(3)] does not appear. Therefore,

n	1	2	≥ 3
$\#\mathfrak{I}(\mathbb{G}^{2n+1}_\star)$	16	28	$10n+9$

We can now state the main theorem of this paper, which solve the classification problem for odd-order tensor spaces.

Theorem 2.6 *Let consider* $\mathbb{T}^{2n+1}_c$ *the space of coupling tensors between two physics described respectively by two tensor vector spaces* $\mathbb{E}_1$ *and* $\mathbb{E}_2$*. If these tensor spaces are of orders greater or equal to* 1*, then*

- $\mathfrak{I}(\mathbb{T}^{2n+1}_c) = \mathfrak{I}(\mathbb{G}^{2n+1}_\star)$ *if* $\mathbb{E}_1 = \mathbb{I}^{2p+1}$ *(resp.* $\mathbb{E}_2 = \mathbb{I}^{2p+1}$*) is a space of odd-order tensors which harmonic decomposition only contains odd-order terms, and* $\mathbb{E}_2 = \mathbb{P}^{2q}$ *(resp.* $\mathbb{E}_1 = \mathbb{P}^{2q}$*) a space of even-order tensors which harmonic decomposition only contains even-order terms;*
- $\mathfrak{I}(\mathbb{T}^{2n+1}_c) = \mathfrak{I}(\mathbb{G}^{2n+1})$ *otherwise.*

It can be noted that the first situation always occurs when $\mathbb{T}_c$ is constructed by the tensor product of two completely symmetric tensor spaces. For example the piezoelectricity is a map between $\mathbb{E}_1 = \mathbb{T}_{(ij)}$, which is a completely symmetric tensor space, and $\mathbb{E}_2 = \mathbb{T}_k$ which is also completely symmetric. Therefore $\mathfrak{I}(\mathbb{P}\mathrm{iez}) = \mathfrak{I}(\mathbb{G}^3_\star)$:

$$\mathfrak{I}(\mathbb{P}\mathrm{iez}) = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{Z}_3], [\mathrm{D}^v_2], [\mathrm{D}^v_3], [\mathrm{Z}^-_2], [\mathrm{Z}^-_{4n}], [\mathrm{D}_2], [\mathrm{D}_3], [\mathrm{D}^h_4], [\mathrm{D}^h_6], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathcal{O}^-], [\mathrm{O}(3)]\}$$

And we therefore retrieve the result of Weller [34]. But some exceptions can appear, for example, the space of elasticity tensors is not completely symmetric but only contains even-order harmonics [14, 28]. Therefore its tensor product with, for example, $\mathbb{T}_i$ will produce a fifth-order tensors space which classes will be the same as $\mathbb{G}^5_\star$.

2.5 Second strain-gradient elasticity

Application of the former theorem will be made on the odd-order tensors of SSGE. First, the constitutive equations will be summed-up, and then the results will be stated. It worth noting that obtaining the same results with the Forte-Vianello approach would have been far more complicated.

Constitutive laws

In the second strain-gradient theory of linear elasticity [13, 27], the constitutive law gives the symmetric Cauchy stress tensor[12] $\sigma^{(2)}$ and the hyperstress tensors $\tau^{(3)}$ and $\omega^{(4)}$ in terms of the infinitesimal strain

[11] There is two reasons why [O(3)] appears as a member of the planar collection. First, [O(3)] is the limit class of $[\mathrm{Z}^-_{2k}]$ for k greater than the tensor order. Second, it ensures the algebraical coherence of the computation made in section 5.

[12] In this subsection only, tensor orders will be indicated by superscripts in parentheses.

tensor $\varepsilon^{(2)}$ and its gradients $\eta^{(3)} = \varepsilon^{(2)} \otimes \nabla$ and $\kappa^{(4)} = \varepsilon^{(2)} \otimes \nabla \otimes \nabla$ through the three linear relations:

$$\begin{cases} \sigma^{(2)} = \mathbf{E}^{(4)} : \varepsilon^{(2)} + \mathbf{M}^{(5)} \therefore \eta^{(3)} + \mathbf{N}^{(6)} :: \kappa^{(4)}, \\ \tau^{(3)} = \mathbf{M}^{T(5)} : \varepsilon + \mathbf{A}^{(6)} \therefore \eta^{(3)} + \mathbf{O}^{(7)} :: \kappa^{(4)}, \\ \omega^{(4)} = \mathbf{N}^{T(6)} : \varepsilon^{(2)} + \mathbf{O}^{T(7)} \therefore \eta^{(3)} + \mathbf{B}^{(8)} :: \kappa^{(4)} \end{cases} \tag{2}$$

where $:, \therefore, ::$ denote, respectively, the double, third and fourth contracted products. Above[13], $\sigma_{(ij)}$, $\varepsilon_{(ij)}$, $\tau_{(ij)k}$, $\eta_{(ij)k} = \varepsilon_{(ij),k}$, $\omega_{(ij)(kl)}$ and $\kappa_{(ij)(kl)} = \varepsilon_{(ij),(kl)}$ are, respectively, the matrix components of $\varepsilon^{(2)}$, $\sigma^{(2)}$, $\tau^{(3)}$, $\eta^{(3)}$, $\omega^{(4)}$ and $\kappa^{(4)}$ relative to an orthonormal basis $(\mathbf{i};\mathbf{j};\mathbf{k})$ of $\mathbb{R}^3$. And $E_{\underline{(ij)}\ \underline{(lm)}}$, $M_{(ij)(lm)n}$, $N_{(ij)(kl)(mn)}$, $A_{\underline{(ij)k}\ \underline{(lm)n}}$, $O_{(ij)k(lm)(no)}$ and $B_{\underline{(ij)(kl)}\ \underline{(mn)(op)}}$ are the matrix components of the related elastic stiffness tensors.

Symmetry classes

The symmetry classes of the elasticity tensors and of the first strain-gradient elasticity tensors has been studied in [14] and [22]. The extension to the even-order tensors of Mindlin SSGE has been considered in the first part of this article [28]. Hence, here, we solely consider the spaces of coupling tensors $\mathbf{M}^{(5)}$ and $\mathbf{O}^{(7)}$.

• We define $\mathbb{C}$ef to be the space of coupling tensors between classical elasticity and first strain-gradient elasticity:

$$\mathbb{C}\mathrm{ef} := \{\mathbf{M}^{(5)} \in \mathbb{G}^5 | M_{(ij)(kl)m}\}$$

This constitutive space is generated by the following tensor product: $\mathbb{C}\mathrm{ef} = \mathbb{T}_{(ij)} \otimes \mathbb{T}_{(kl)m}$. Using the Clebsch-Gordan product[14] we obtain : $\mathbb{T}_{(kl)m} = \mathbb{H}^3 \oplus \mathbb{H}^{2*} \oplus 2\mathbb{H}^1$, therefore an even-order tensor is contained in the harmonic decomposition of $\mathbb{T}_{(kl)m}$. And hence, a direct application of Theorem 2.6 leads to the result:

$$\begin{aligned} \mathfrak{I}(\mathbb{C}\mathrm{ef}) = \mathfrak{I}(\mathbb{G}^5) \quad = \quad & \{[\mathbb{1}], [\mathrm{Z}_2], \cdots, [\mathrm{Z}_5], [\mathrm{D}_2^v], \cdots, [\mathrm{D}_5^v], [\mathrm{Z}_2^-], \cdots, [\mathrm{Z}_8^-], [\mathrm{D}_2], \cdots, [\mathrm{D}_5], \\ & [\mathrm{D}_4^h], \cdots, [\mathrm{D}_{10}^h], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathcal{T}], [\mathcal{O}^-], [\mathcal{O}], \mathrm{SO}(3), \mathrm{O}(3)\} \end{aligned}$$

Therefore $\mathbb{C}$ef is divided into 29 symmetry classes.

• We define $\mathbb{C}$fs to be the space of coupling tensors between first strain-gradient elasticity and second strain-gradient elasticity:

$$\mathbb{C}\mathrm{fs} = \{\mathbf{O}^{(7)} \in \mathbb{G}^7 | O_{(ij)k(lm)(no)}\}$$

This constitutive space is generated by the following tensor product: $\mathbb{C}\mathrm{fs} = \mathbb{T}_{(ij)k} \otimes \mathbb{T}_{(lm)(no)}$. Therefore a direct application of Theorem 2.6 leads to the result:

$$\begin{aligned} \mathfrak{I}(\mathbb{C}\mathrm{fs}) = \mathfrak{I}(\mathbb{G}^7) \quad = \quad & \{[\mathbb{1}], [\mathrm{Z}_2], \cdots, [\mathrm{Z}_7], [\mathrm{D}_2^v], \cdots, [\mathrm{D}_7^v], [\mathrm{Z}_2^-], \cdots, [\mathrm{Z}_{12}^-], [\mathrm{D}_2], \cdots, [\mathrm{D}_7], \\ & [\mathrm{D}_4^h], \cdots, [\mathrm{D}_{14}^h], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathcal{T}], [\mathcal{O}^-], [\mathcal{O}], [\mathcal{I}], \mathrm{SO}(3), \mathrm{O}(3)\} \end{aligned}$$

Hence $\mathbb{C}$fs is divided into 40 symmetry classes.

3 General framework

In this section, the mathematical framework of symmetry analysis is introduced. In the first two subsections the notions of symmetry group and class are introduced; the last ones are devoted to the definition of irreducible spaces. The presentation is rather general, and will be specialized to tensor spaces only at the end of the section.

[13] The comma classically indicates the partial derivative with respect to spatial coordinates. Superscript T denotes transposition. The transposition is defined by permuting the p first indices with the q last, where p is the tensorial order of the image of a q-order tensor.

[14] see subsection 3.4 for details

3.1 Isotropy/symmetry groups

Let ρ be a linear representation of a compact real Lie group[15] G on a finite dimensional $\mathbb{R}$-linear space $\mathcal{E}$:

$$\rho : G \to \mathrm{GL}(\mathcal{E})$$

This action will also be denoted

$$g \cdot \mathbf{x} = \rho(g)(\mathbf{x})$$

where $g \in G$ and $\mathbf{x} \in \mathcal{E}$. For any element $\mathbf{x}$ of $\mathcal{E}$, the set of operations g in G leaving this element invariant is defined as

$$\Sigma_{\mathbf{x}} = \{g \in G \mid g \cdot \mathbf{x} = \mathbf{x}\}$$

This set is known to physicists as the symmetry group of $\mathbf{x}$ and to mathematicians as the stabilizer or the isotropy subgroup of $\mathbf{x}$. Owing to G-compactness, every isotropy subgroup is a closed subgroup of G (see [8]). Conversely, a dual notion can be defined for G-elements. For any subgroup K of G, the set of K-invariant elements in $\mathcal{E}$ is defined as

$$\mathcal{E}^K = \{\mathbf{x} \in \mathcal{E} \mid \ k \cdot \mathbf{x} = \mathbf{x} \text{ for all } k \in K\}$$

Such a set is referred to as a fixed point set and is a linear subspace of $\mathcal{E}$.

3.2 Isotropy/symmetry classes

We aim to describe objects that have the same symmetry properties but may differ by their orientations in space. The first point is to define the set of all the positions an object can have. To that aim we consider the G-orbit of an element $\mathbf{x}$ of $\mathcal{E}$:

$$\mathrm{Orb}(\mathbf{x}) = \{g \cdot \mathbf{x} \mid g \in G\} \subset \mathcal{E}$$

Due to G-compactness this set is a submanifold of $\mathcal{E}$ (see [8]). Elements of $\mathrm{Orb}(\mathbf{x})$ will be said to be G-related. A fundamental observation is that G-related vectors have conjugate symmetry groups. More precisely,

$$\mathrm{Orb}(\mathbf{x}) = \mathrm{Orb}(\mathbf{y}) \Rightarrow \Sigma_{\mathbf{x}} = g\Sigma_{\mathbf{y}}g^{-1} \text{ for some } g \in G$$

Let us define the conjugacy class of a subgroup $K \subset G$ by

$$[K] = \{K' \subset G | K' = gKg^{-1} \text{ for some } g \in G\}$$

An isotropy class (or symmetry class) $[\Sigma]$ is defined as the conjugacy class of an isotropy subgroup Σ. This definition implies that there exists a vector $\mathbf{x} \in \mathcal{E}$ such that $\Sigma_{\mathbf{x}} \in [\Sigma]$; meaning $\Sigma_{\mathbf{x}} = g\Sigma g^{-1}$ for some $g \in G$. The notion of isotropy class is the good notion to define the symmetry property of an object modulo its orientation. Due to G-compactness there is only a finite number of isotropy classes [8], and we introduce the notation

$$\mathfrak{I}(\mathcal{E}) := \{[\mathbb{1}]; [\Sigma_1]; \cdots ; [\Sigma_l]\}$$

to denote the set of all isotropy classes. In the case $G = \mathrm{O}(3)$ this result is known as the Hermann's theorem [2, 19]. The elements of $\mathfrak{I}(\mathcal{E})$ are classes of subgroups conjugate to $\mathrm{O}(3)$-closed subgroups; this collection was introduced in subsection 2.4.

3.3 Irreducible spaces

For every linear subspace $\mathcal{F}$ of $\mathcal{E}$, we set

$$g \cdot \mathcal{F} := \{g.\mathbf{x} \mid g \in G\, ;\, \mathbf{x} \in \mathcal{F}\}$$

and we say that $\mathcal{F}$ is G-stable if $g \cdot \mathcal{F} \subset \mathcal{F}$ for every $g \in G$. It is clear that, for every representation, the subspaces $\{0\}$ and $\mathcal{E}$ are always G-stable. If, for a representation ρ on $\mathcal{E}$, the only G-invariant spaces are the proper ones, the representation will be said to be irreducible. For a compact Lie group, the

[15] In the following G will always represent a compact real Lie group, so this specification will mostly be omitted.

Peter-Weyl theorem [30] ensures that every representation can be split into a direct sum of irreducible ones. Furthermore, in the case $G = \mathrm{SO}(3)$, those irreducible representations are explicitly known [35]. We will then be able to deduce the O(3)-irreducible representations.

Let's first recall that there is a natural SO(3) representation on the space of $\mathbb{R}^3$-harmonic polynomials[16]. If p is a harmonic polynomial, and $\mathbf{x} \in \mathbb{R}^3$, then for every $g \in \mathrm{SO}(3)$ we write

$$g \cdot p(\mathbf{x}) = p(g^{-1} \cdot \mathbf{x})$$

Harmonic polynomials form a graded algebra, and to each subspace of a given degree is associated an SO(3)-irreducible representation. $\mathcal{H}^k$ will be the vector space of harmonic polynomials of degree k, with $\dim \mathcal{H}^k = 2k+1$. If we take a vector space V to be a SO(3)-representation, it can be decomposed into SO(3)-irreducible spaces[17] [30]

$$V = \bigoplus \mathcal{H}^{k_i}$$

Grouping together irreducible spaces of the same order, one obtains the SO(3)-isotypic decomposition of a representation:

$$V = \bigoplus_{i=0}^{n} \alpha_i \mathcal{H}^i$$

where α_i is the multiplicity of the irreducible space $\mathcal{H}^i$ in the decomposition, and n is the order of the highest-order irreducible space of the decomposition.

3.4 Application to tensor spaces: the harmonic decomposition

In mechanics, V is a vector subspace of $\otimes^p \mathbb{R}^3$. Thus, we have a natural O(3)-linear action on this subspace; but it is clear that, for every $\mathbf{T} \in \otimes^p \mathbb{R}^3$ and for every g in O(3) we will have

$$g \cdot \mathbf{T} = (\det(g))^p \times (\det(g) g) \cdot \mathbf{T} \tag{3}$$

For the case where p is even (as, for example, in the case of linear elasticity), this O(3)-action can therefore be reduced to the case of an SO(3)-linear action, and we can deduce an irreducible decomposition given by **??**. But one can use a classical isomorphism[18] in $\mathbb{R}^3$ between the space of harmonic polynomials of degree i and the space of i-th order completely symmetric traceless tensors.

Now, in the case where p is odd, the same equality 3 shows that the natural O(3)-linear action on $\otimes^p \mathbb{R}^3$ can be obtained from the SO(3)-linear action; thus the same argument as above shows that we have an irreducible decomposition with the subspaces $\mathbb{H}^k$, but the associated linear O(3)-action is either ρ_k, or $\rho_k^* := \det(g)\rho_k$. We will denote $\mathbb{H}^k$ for the ρ_k action and $\mathbb{H}^{k*}$ for the ρ_k^* action. As an example, we know that in the case of piezoelasticity, the associated irreducible decomposition is $\mathbb{H}^3 \oplus \mathbb{H}^{2*} \oplus 2\mathbb{H}^1$ [34]. We can then have a decomposition

$$V = \bigoplus_{i=0}^{p} \alpha_i \mathbb{V}^i \text{ where } \mathbb{V}^i = \mathbb{H}^i \text{ or } \mathbb{H}^{i*} \tag{4}$$

Nevertheless, to simplify notations, we will not specify distinction between the two representations ρ_k and ρ_k^*, and we will note

$$V = \bigoplus_{i=0}^{p} \alpha_i \mathbb{H}^i$$

being aware that in the case of an even harmonic tensor space, we do not have the standard O(3)-linear action.

[16] A polynomial is said to be harmonic if its Laplacian is null.
[17] we recall here that equality means isomorphism
[18] In [5] Backus investigates this isomorphism and shows that this association is no more valid in $\mathbb{R}^{N \geq 4}$.

The symmetry group of $\mathbf{T} \in \mathbb{T}^n$ is the intersection of the symmetry groups of all its harmonic components[19]:

$$\Sigma_{\mathbf{T}} = \bigcap_{i=0}^{n} \left(\bigcap_{j=0}^{\alpha_i} \Sigma_{\mathrm{H}^{i,j}} \right)$$

The object of the next section is the determination of the symmetry class of an element $\mathbf{T} \in V$ from the symmetry classes of its components. The symmetry classes of O(3)-irreducible representations are explicitly known [17, 20], what is unknown is how to combine these results to determine the symmetry classes of V.

4 The clips operation

The aim of this section is to construct symmetry classes of a reducible representation from irreducible ones. For that goal, and as in [28], a class-operator, named the clips operator, is defined. The main results of this section are given in tables 1 and 2, which contain all clips operations between SO(3)-closed subgroups and O(3)-closed subgroups. The explicit proofs of these results can be found in [28] for SO(3) and some sketches of the proofs are in the Appendix A for O(3).

Let us start with the following lemma:

Lemma 4.1 *Let $\mathcal{E}$ be a representation of a compact Lie group G that splits into a direct sum of two G-stable subspaces:*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \text{ where } g \cdot \mathcal{E}_1 \subset \mathcal{E}_1 \text{ and } g \cdot \mathcal{E}_2 \subset \mathcal{E}_2 \, \forall g \in G$$

If we denote by $\mathfrak{I}$ the set of all isotropy classes associated with $\mathcal{E}$ and by $\mathfrak{I}_i$ the set of all isotropy classes associated with $\mathcal{E}_i$ $(i = 1, 2)$, then $[\Sigma] \in \mathfrak{I}$ if and only if there exist $[\Sigma_1] \in \mathfrak{I}_1$ and $[\Sigma_2] \in \mathfrak{I}_2$ such that $\Sigma = \Sigma_1 \cap \Sigma_2$.

Proof. If we take $[\Sigma_1] \in \mathfrak{I}_1$ and $[\Sigma_2] \in \mathfrak{I}_2$, we know there exist two vectors $\mathbf{x}_1 \in \mathcal{E}_1$ and $\mathbf{x}_2 \in \mathcal{E}_2$ such that $\Sigma_i = \Sigma_{\mathbf{x}_i}$ $(i = 1, 2)$. Then, let $\mathbf{x} := \mathbf{x}_1 + \mathbf{x}_2$.

For every $g \in \Sigma_1 \cap \Sigma_2$ we have $g \cdot \mathbf{x}_1 + g \cdot \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}$; thus $\Sigma_1 \cap \Sigma_2 \subset \Sigma_{\mathbf{x}}$. Conversely for every $g \in \Sigma_{\mathbf{x}}$ we have

$$g \cdot \mathbf{x} = \mathbf{x} = g \cdot \mathbf{x}_1 + g \cdot \mathbf{x}_2$$

But, since the $\mathcal{E}_i$ are G-stable and form direct sum, we conclude that $g \cdot \mathbf{x}_i = \mathbf{x}_i$ $(i = 1, 2)$. The reverse inclusion is proved.

The other implication is similar: if we take $[\Sigma] \in \mathfrak{I}$ then we have $\Sigma = \Sigma_{\mathbf{x}}$ for some $\mathbf{x} \in \mathcal{E}$ and $\mathbf{x}$ can be decomposed into $\mathbf{x}_1 + \mathbf{x}_2$. The same proof as above shows that $\Sigma = \Sigma_{\mathbf{x}_1} \cap \Sigma_{\mathbf{x}_2}$. □

Lemma 4.1 shows that the isotropy classes of a direct sum are related to intersections of isotropy subgroups. But as intersection of classes is meaningless, the results cannot be directly extended. To solve this problem, a tool called the clips operator will be introduced. We will make sure of a lemma:

Lemma 4.2 *For every two G-classes $[\Sigma_i]$ $(i = 1, 2)$, and for every g_1, g_2 in G, there exists $g = g_1^{-1} g_2$ in G such that*

$$\left[g_1 \Sigma_1 g_1^{-1} \cap g_2 \Sigma_2 g_2^{-1}\right] = \left[\Sigma_1 \cap g \Sigma_2 g^{-1}\right]$$

Proof. Let $g = g_1^{-1} g_2$ and

$$\Sigma = g_1 \Sigma_1 g_1^{-1} \cap g_2 \Sigma_2 g_2^{-1}$$

For every $\gamma \in \Sigma$ we have $\gamma = g_1 \gamma_1 g_1^{-1} = g_2 \gamma_2 g_2^{-1}$ for some $\gamma_i \in \Sigma_i$ $(i = 1, 2)$; then

$$g_1 \gamma g_1^{-1} = \gamma_1 \in \Sigma_1 \text{ and } g_1 \gamma g_1^{-1} = g \gamma_2 g^{-1} \in g \Sigma_2 g^{-1}$$

Thus we have $g_1 \Sigma g_1^{-1} \subset \Sigma_1 \cap g \Sigma_2 g^{-1}$, and conversely. As $g_1 \Sigma g_1^{-1}$ is conjugate to Σ, we have proved the lemma. □

[19] In the notation $\mathrm{H}^{i,j}$ the first superscript refers to the order of the harmonic tensor, while the second indexes the multiplicity of H^i in the decomposition.

Definition 4.3 (Clips Operator) We define the action of the clips operator $\circledcirc$ on G-classes $[\Sigma_1]$ and $[\Sigma_2]$ by setting

$$[\Sigma_1] \circledcirc [\Sigma_2] := \{[\Sigma_1 \cap g\Sigma_2 g^{-1}] \quad \text{for all } g \in G\}$$

which is a subset of G-classes.

The idea of this clips operator is simply to get, step by step, all possible intersections between closed SO(3)- and O(3)-subgroups. Arguing on axes, we are able to determine all possible intersections, and gave the results with this class operator. One can observe some immediate properties:

Proposition 4.4 *For every G-class $[\Sigma]$ we have*

$$[\mathbb{1}] \circledcirc [\Sigma] = \{[\mathbb{1}]\} \text{ and } [G] \circledcirc [\Sigma] = \{[\Sigma]\}$$

Given two G-representations $\mathcal{E}_1$ and $\mathcal{E}_2$, and if we denote $\mathfrak{I}_i$ the set of all isotropy classes of E_i, the action of the clips operator can be extended to these sets via

$$\mathfrak{I}_1 \circledcirc \mathfrak{I}_2 := \bigcup_{\Sigma_1 \in \mathfrak{I}, \Sigma_2 \in \mathfrak{I}_2} [\Sigma_1] \circledcirc [\Sigma_2]$$

We have the property [28]:

Proposition 4.5 *For every two G-representations $\mathcal{E}_1$ and $\mathcal{E}_2$, if $\mathfrak{I}_1$ denotes the isotropy classes of $\mathcal{E}_1$ and $\mathfrak{I}_2$ the isotropy classes of $\mathcal{E}_2$, then $\mathfrak{I}_1 \circledcirc \mathfrak{I}_2$ are all the isotropy classes of $\mathcal{E}_1 \oplus \mathcal{E}_2$.*

Our main results concerning the clips operators are:

Theorem 4.6 *For every two* SO(3)*-closed subgroups Σ_1 and Σ_2, we have $\mathbb{1} \in [\Sigma_1] \circledcirc [\Sigma_2]$. The remaining classes in the clips product $[\Sigma_1] \circledcirc [\Sigma_2]$ are given in the table 1.*

Theorem 4.7 *For every two closed subgroups Σ_1 and Σ_2 of* O(3)*, with $[\Sigma_1]$ or $[\Sigma_2]$ distinct from $[\mathrm{O}(2)^-]$, we always have $\mathbb{1} \in [\Sigma_1] \circledcirc [\Sigma_2]$. For every closed subgroup Σ of* SO(3)*:*

$$\begin{array}{rcl} [\mathrm{Z}_2^-] \circledcirc [\Sigma] = [\mathbb{1}] & \quad & [\mathrm{Z}_{2n}^-] \circledcirc [\Sigma] = [\mathrm{Z}_n] \circledcirc [\Sigma] \\ [\mathrm{D}_n^v] \circledcirc [\Sigma] = [\mathrm{Z}_n] \circledcirc [\Sigma] & \quad & [\mathrm{D}_{2n}^h] \circledcirc [\Sigma] = [\mathrm{D}_n] \circledcirc [\Sigma] \\ [\mathcal{O}^-] \circledcirc [\Sigma] = [\mathcal{T}] \circledcirc [\Sigma] & \quad & [\mathrm{O}(2)^-] \circledcirc [\Sigma] = [\mathrm{SO}(2)] \circledcirc [\Sigma] \end{array}$$

Clips operation $[\Sigma_1] \circledcirc [\Sigma_2]$ for type III subgroups are given in the table 2.

Table 1 Clips operations on SO(3)-subgroups

$\circledcirc$	$[\mathrm{Z}_n]$	$[\mathrm{D}_n]$	$[\mathcal{T}]$	$[\mathcal{O}]$	$[\mathcal{I}]$	$[\mathrm{SO}(2)]$	$[\mathrm{O}(2)]$
$[\mathrm{Z}_m]$	$[\mathrm{Z}_d]$						
$[\mathrm{D}_m]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_d]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_{d'_2}]$, $[\mathrm{Z}_{dz}]$ $[\mathrm{Z}_d]$, $[\mathrm{D}_d]$					
$[\mathcal{T}]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_{d_3}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_{d_3}]$, $[\mathrm{D}_{d_2}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$ $[\mathcal{T}]$				
$[\mathcal{O}]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_{d_3}]$ $[\mathrm{Z}_{d_4}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_{d_3}]$, $[\mathrm{Z}_{d_4}]$ $[\mathrm{D}_{d_2}]$, $[\mathrm{D}_{d_3}]$ $[\mathrm{D}_{d_4}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$ $[\mathcal{T}]$	$[\mathrm{Z}_2]$ $[\mathrm{D}_2]$, $[\mathrm{Z}_3]$ $[\mathrm{D}_3]$, $[\mathrm{Z}_4]$ $[\mathrm{D}_4]$, $[\mathcal{O}]$			
$[\mathcal{I}]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_{d_3}]$ $[\mathrm{Z}_{d_5}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_{d_3}]$, $[\mathrm{Z}_{d_5}]$ $[\mathrm{D}_{d_2}]$ $[\mathrm{D}_{d_3}]$, $[\mathrm{D}_{d_5}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$ $[\mathcal{T}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$, $[\mathrm{D}_3]$ $[\mathcal{T}]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$, $[\mathrm{D}_3]$ $[\mathrm{Z}_5]$, $[\mathrm{D}_5]$ $[\mathcal{I}]$		
$[\mathrm{SO}(2)]$	$[\mathrm{Z}_n]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_n]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$, $[\mathrm{Z}_4]$	$[\mathrm{Z}_2]$ $[\mathrm{Z}_3]$, $[\mathrm{Z}_5]$	$[\mathrm{SO}(2)]$	
$[\mathrm{O}(2)]$	$[\mathrm{Z}_{d_2}]$ $[\mathrm{Z}_n]$	$[\mathrm{Z}_2]$ $[\mathrm{D}_n]$	$[\mathrm{D}_2]$ $[\mathrm{Z}_3]$	$[\mathrm{D}_2]$ $[\mathrm{D}_3]$, $[\mathrm{D}_4]$	$[\mathrm{D}_2]$ $[\mathrm{D}_3]$, $[\mathrm{D}_5]$	$[\mathrm{Z}_2]$ $[\mathrm{SO}(2)]$	$[\mathrm{Z}_2]$ $[\mathrm{O}(2)]$

Notations

$\mathrm{Z}_1 := \mathrm{D}_1 := \mathbb{1}$ $\quad d_2 := gcd(n,2)$ $\quad d_3 := gcd(n,3)$ $\quad d_5 := gcd(n,5)$

$d'_2 := gcd(m,2)$ $\quad dz := 2$ if $d=1$, $dz = 1$ otherwise

$$d_4 := \begin{cases} 4 \text{ if } 4 \mid n \\ 1 \text{ otherwise} \end{cases}$$

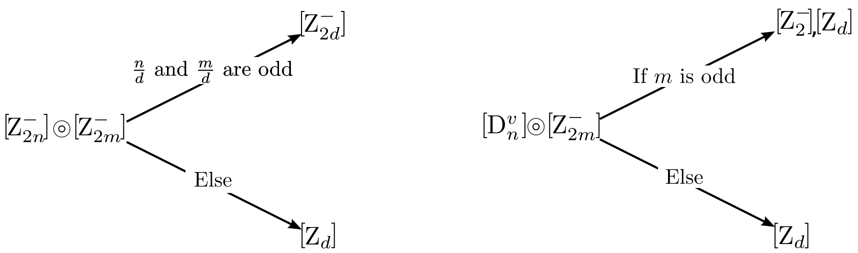

Fig. 1

Table 2 Clips operations on type III O(3)-subgroups

$\circledcirc$	$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_{2m}^-]$	$[\mathrm{D}_m^v]$	$[\mathrm{D}_{2m}^h]$	$[\mathcal{O}^-]$	$[\mathrm{O}(2)^-]$
$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_2^-]$					
$[\mathrm{Z}_{2n}^-]$	$[\mathrm{Z}_{i(n)}^-]$	Figure 1				
$[\mathrm{D}_n^v]$	$[\mathrm{Z}_2^-]$	Figure 1	$[\mathrm{Z}_2^-]$, $[\mathrm{D}_d^v]$ $[\mathrm{Z}_d]$			
$[\mathrm{D}_{2n}^h]$	$[\mathrm{Z}_2^-]$	Figure 2	Figure 3	Figure 4		
$[\mathcal{O}^-]$	$[\mathrm{Z}_2^-]$	Figure 5	$[\mathrm{Z}_2^-]$ $[\mathrm{Z}_{d_3(m)}]$ $[\mathrm{D}_{d_3(m)}^v]$ $[\mathrm{Z}_{d_2(m)}]$ $[\mathrm{Z}_{d_2(m)}^v]$	Figure 6	$[\mathrm{Z}_2^-]$, $[\mathrm{Z}_4^-]$ $[\mathrm{Z}_3]$	
$[\mathrm{O}(2)^-]$	$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_{i(m)}^-]$ $[\mathrm{Z}_m]$	$[\mathrm{Z}_2^-]$, $[\mathrm{D}_m^v]$	$[\mathrm{Z}_{i(m)}]$, $[\mathrm{Z}_2^-]$ $[\mathrm{D}_{i(m)}^v]$, $[\mathrm{D}_m^v]$	$[\mathrm{Z}_2^-]$, $[\mathrm{D}_3^v]$ $[\mathrm{D}_2^v]$	$[\mathrm{Z}_2^-]$, $[\mathrm{O}(2)^-]$

Notations for table 2:

$\mathrm{Z}_1^- := \mathrm{D}_1^v := \mathbb{1}$ $\quad d = gcd(n,m) \quad d_2(n) = gcd(n,2)$

$i(n) := 3 - gcd(2,n) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 2 \text{ if } n \text{ is odd} \end{cases} \quad d_3(n) = gcd(n,3)$

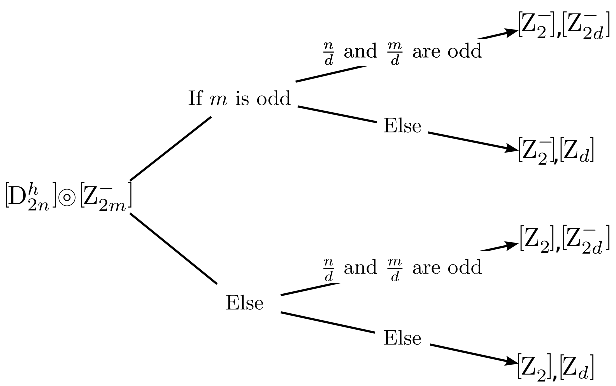

Fig. 2

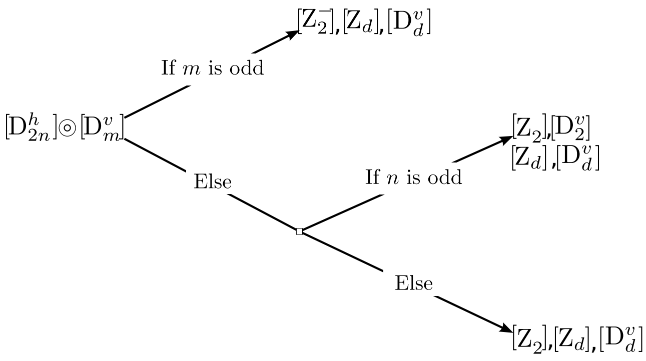

Fig. 3

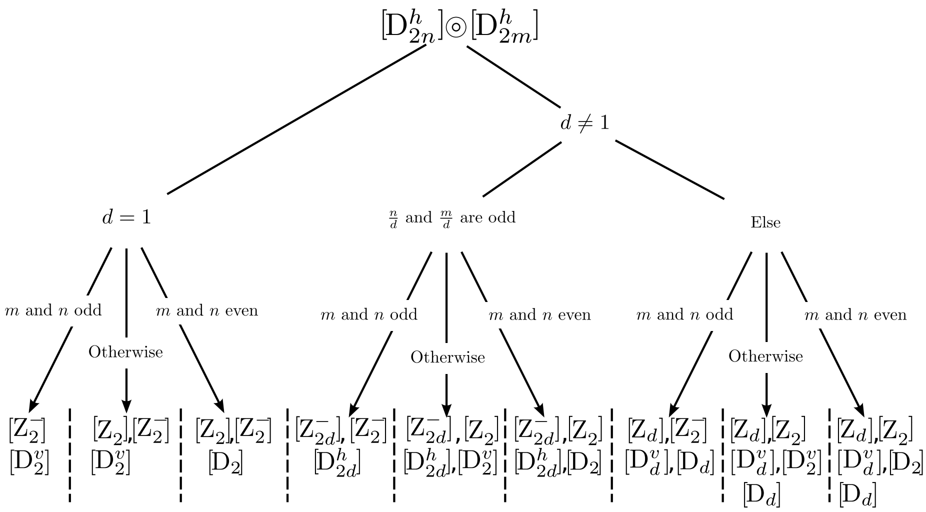

Fig. 4

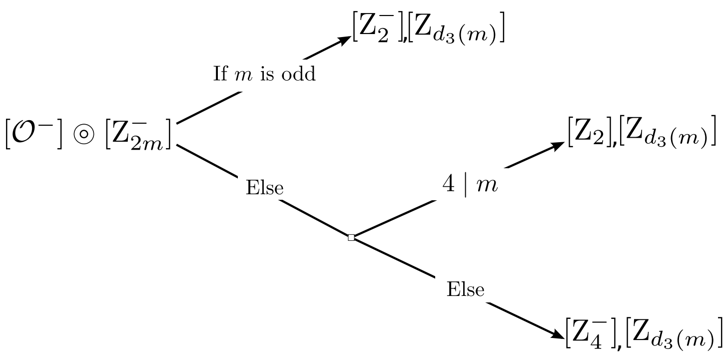

Fig. 5

5 Isotropy classes of constitutive tensors

We now turn to the construction of the symmetry classes of a reducible representation from its irreducible components. The first subsection states the main results on symmetry classes of irreducible representations. Thereafter we derive from the results of the previous section the basic properties of reducible representations. These results will be used to prove the theorem stated in section 2.

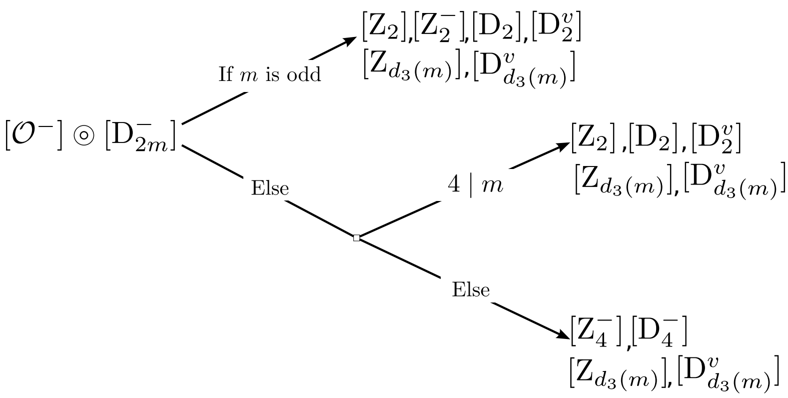

Fig. 6

5.1 Isotropy classes of irreducibles

The following result was obtained by Golubitsky and al. [17, 20]:

Theorem 5.1 *Let* $\mathrm{O}(3)$ *act on* $\mathbb{H}^{k>0}$. *The following groups are symmetry classes of* $\mathbb{H}^k$ *:*

(a) $\mathbb{1}$ *for* $k \geq 3$*;*

(b) Z_n *for* $2 \leq n \leq \dfrac{k}{2}$*;*

(c) Z_{2n}^- *for* $n \leq \dfrac{k}{3}$*;*

(d) D_n *for* $\begin{cases} 1 < n \leq \dfrac{k}{2} \text{ when } k \text{ is odd} \\ 1 < n \leq k \text{ when } k \text{ is even} \end{cases}$

(e) D_n^v *for* $\begin{cases} 1 < n \leq k \text{ when } k \text{ is odd} \\ 1 < n \leq \dfrac{k}{2} \text{ when } k \text{ is even} \end{cases}$

(f) D_{2n}^h *for* $1 < n \leq k$*, except* D_4^h *for* $k = 3$*;*

(g) $\mathcal{T}$ *for* $k \neq 1, 2, 3, 5, 7, 8, 11$*;*

(h) $\mathcal{O}$ *for* $k \neq 1, 2, 3, 5, 7, 11$*;*

(i) $\mathcal{O}^-$ *for* $k \neq 1, 2, 4, 5, 8$*;*

(j) $\mathcal{I}$ *for* $k = 6, 10, 12, 15, 16, 18$ *or* $l \geq 20$ *and* $l \neq 23, 29$*;*

(k) $\mathrm{O}(2)$ *for* k *even;*

(l) $\mathrm{O}(2)^-$ *for* k *odd.*

Now everything is in order to construct the proof of our theorem. Therefore it is the aim of the remaining part of this paper

5.2 Symmetry classes of odd-order tensor spaces

Let us consider the constitutive tensor space $\mathbb{T}^{2n+1}$. It is known that this space can be decomposed orthogonally into a full symmetric space and a complementary one which is isomorphic to a tensor space of order $2n$ [21], i.e.:

$$\mathbb{T}^{2n+1} = \mathbb{S}^{2n+1} \oplus \mathbb{C}^{2n}$$

Let us consider the O(3)-isotypic decomposition of $\mathbb{T}^{2n+1}$

$$\mathbb{T}^{2n+1} = \bigoplus_{k=0}^{2n+1} \alpha_k \mathbb{H}^k, \quad \text{with } \alpha_{2n+1} = 1$$

The part related to $\mathbb{S}^{2n+1}$ solely contains odd-order harmonic tensors with multiplicity one [21], that is,

$$\mathbb{S}^{2n+1} = \bigoplus_{k=0}^{n} \mathbb{H}^{2k+1} \quad \text{and} \quad \mathbb{C}^{2n} = \bigoplus_{k=0}^{2n} \alpha'_k \mathbb{H}^k \quad \text{with } \alpha'_k = \begin{cases} \alpha_k - 1 \text{ for } k \text{ odd} \\ \alpha_k \text{ for } k \text{ even} \end{cases}$$

5.2.1 Symmetric tensor space

Let us first determine the symmetry classes of $\mathbb{S}^{2n+1}$. We have $\mathfrak{I}(\mathbb{S}^{2n+1}) = \mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) \cup \mathcal{S}(\mathfrak{I}(\mathbb{S}^{2n+1}))$, and $\mathcal{S}(\mathfrak{I}(\mathbb{S}^{2n+1}))$ can be directly obtained:

Lemma 5.2 *The spatial symmetry classes of* $\mathbb{S}^{2n+1}$ *are:*

n	*1-3*	*4-6*	$\geq$ *7*
$\mathcal{S}(\mathfrak{I}(\mathbb{S}^{2n+1}))$	$\{[\mathcal{O}^-]\}$	$\{[\mathcal{T}],[\mathcal{O}^-],[\mathcal{O}]\}$	$\{[\mathcal{T}],[\mathcal{O}^-],[\mathcal{O}],[\mathcal{I}]\}$

Proof. It is a direct application of the theorem 5.1 and the harmonic decomposition of $\mathbb{S}^{2n+1}$. □

It remains to study $\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1}))$. The answer lies in the following theorem

Lemma 5.3 *The planar symmetry classes of* $\mathbb{S}^{2n+1}$ *are:*

$$\begin{aligned} \mathcal{P}(\mathfrak{I}(\mathbb{S}^{3})) &= \{[\mathbb{1}],[\mathrm{D}_2^v],[\mathrm{D}_3^v],[\mathrm{Z}_2^-],[\mathrm{D}_6^h],[\mathrm{O}(2)^-],[\mathrm{O}(3)]\} \\ \mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) &= \{[\mathbb{1}],[\mathrm{Z}_2],\cdots,[\mathrm{Z}_{2n-1}],[\mathrm{D}_2^v],\cdots,[\mathrm{D}_{2n+1}^v],[\mathrm{Z}_2^-],\cdots,[\mathrm{Z}_{2(2n-1)}^-], \\ &\quad [\mathrm{D}_2],\cdots,[\mathrm{D}_n],[\mathrm{D}_4^h],\cdots,[\mathrm{D}_{2(2n+1)}^h],[\mathrm{O}(2)^-],[\mathrm{O}(3)]\} \end{aligned}$$

With the following cardinality:

$$\forall n \geq 2,\ \#\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) = 9n - 1$$

Proof. For $\mathbb{S}^3$ the result is directly obtained using the theorem 5.1. Since $n \geq 2$ a general result appears. Let us decompose $\mathbb{S}^{2n+1}$ into $\bigoplus_{k=0}^{n} \mathbb{H}^{2k+1}$ and as it will be shown

$$\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) = \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1})) \circledcirc \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n-1}))$$

In other terms, the information on the symmetry classes is carried by the 2 higher-order harmonic spaces. Using theorem 5.1, we have

$$\begin{aligned} \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1})) &= \{[\mathbb{1}],[\mathrm{Z}_2],\cdots,[\mathrm{Z}_n],[\mathrm{D}_2^v],\cdots,[\mathrm{D}_{2n+1}^v],[\mathrm{Z}_2^-],\cdots,[\mathrm{Z}_{2\lfloor \frac{2n+1}{3} \rfloor}^-], \\ &\quad [\mathrm{D}_2],\cdots,[\mathrm{D}_n],[\mathrm{D}_4^h],\cdots,[\mathrm{D}_{2(2n+1)}^h],[\mathrm{O}(2)^-],[\mathrm{O}(3)]\} \end{aligned}$$

In this list the collection of $[\mathrm{D}_k^v]$ and $[\mathrm{D}_{2(2k+1)}^h]$ are complete but the following classes are missing:

$$\{[\mathrm{Z}_{n+1}],\cdots,[\mathrm{Z}_{2n+1}],[\mathrm{SO}(2)],[\mathrm{D}_{n+1}],\cdots,[\mathrm{D}_{2n+1}],[\mathrm{O}(2)],[\mathrm{Z}_{2(\lfloor \frac{2n+1}{3} \rfloor +1)}^-],\cdots[\mathrm{Z}_{2(2n+1)}^-],[\mathrm{SO}(3)]\}$$

The clips of $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1}))$ with $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n-1}))$ generates new classes. More precisely, (see table 2)

- The products $\{[\mathrm{D}_k^v] \circledcirc [\mathrm{D}_k^v]\}_{n+1 \leq k \leq 2n-1}$ generate $\{[\mathrm{Z}_{n+1}],\cdots,[\mathrm{Z}_{2n-1}]\}$;
- The products $\{[\mathrm{D}_{2k}^h] \circledcirc [\mathrm{D}_{2k}^h]\}_{n+1 \leq k \leq 2n-1}$ generate $\{[\mathrm{Z}_{2(\lfloor \frac{2n+1}{3} \rfloor +1)}^-],\cdots[\mathrm{Z}_{2(2n-1)}^-]\}$. but the following

cyclic classes can not be generated in this way:

$$\{[\mathrm{Z}_{2n}],[\mathrm{Z}_{2n+1}],[\mathrm{Z}_{4n}^-],[\mathrm{Z}_{2(2n+1)}^-],[\mathrm{SO}(3)]\}$$

and adding lower odd-order harmonic space does not change the situation. The classes $[\mathrm{SO}(2)]$ and $[\mathrm{O}(2)]$ can only be generated by clips operations with $[\mathrm{O}(2)]$. As the class $[\mathrm{O}(2)]$ appears for even-order harmonic tensor, $[\mathrm{SO}(2)]$ and $[\mathrm{O}(2)]$ cannot be symmetry classes of $\mathbb{S}^{2n+1}$. The same observation can be made concerning $[\mathrm{SO}(3)]$. For $[\mathrm{SO}(3)]$ to appear, a $\mathbb{H}^0$ has to be contained in the decomposition of $\mathbb{S}^{2n+1}$, and this is not the case. It remains to consider the dihedral classes. Inspection of table 2 reveals that dihedral classes can only be generated by $[\mathrm{D}_{2p}^h] \circledcirc [\mathrm{D}_{2q}^h]$ products. The analysis of the figure (4) shows that such an operation generates the following dihedral classes: $[\mathrm{D}_d]$ with $d = gcd(n,m)$. As $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1}))$ contains dihedral classes up to $[\mathrm{D}_n]$, this product is interesting if $[\mathrm{D}_{n+k}]$ can be generated for some k, $1 \leq k \leq n+1$. To that aim, we must have $d = gcd(p,q) = n+k$, such a condition implies that there exists $p', q' \in \mathbb{N}$ such as

$$p = p'(n+k) \quad q = q'(n+k)$$

But, because $p \leq 2n+1$ and $q \leq 2n-1$, we must have $p' = q' = 1$ and therefore $p = q = (n+k)$. But in such a situation $\frac{p}{d} = \frac{q}{d} = 1$, and according to the figure (4) no dihedral class is generated. As a conclusion, the clips operation $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1})) \circledcirc \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n-1}))$ can not generate the missing dihedral classes. And, as adding lower odd-order tensors in the clips operation will not change the situation, the dihedral classes of $\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1}))$ are the same as the ones of $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n+1})) \circledcirc \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n-1}))$. This concludes the proof. $\square$

Summing the results for planar and spatial classes we therefore obtain:

n	1	$2-3$	$4-6$	≥ 7
$\#\mathfrak{I}(\mathbb{S}^{2n+1})$	8	$9n$	$9n+2$	$9n+3$

5.2.2 Generic tensor space

Let us now consider $\mathbb{G}^{2n+1} := \otimes^{2n+1}\mathbb{R}^3$.

Lemma 5.4 *Let $\mathbb{G}^{2n+1}$ be the generic tensor space, its spatial symmetry classes are:*

n	*1*	*2*	$\geq$ *3*
$\mathcal{S}(\mathfrak{I}(\mathbb{G}^{2n+1}))$	$\{[\mathcal{O}^-]\}$	$\{[\mathcal{T}],[\mathcal{O}^-],[\mathcal{O}]\}$	$\{[\mathcal{T}],[\mathcal{O}^-],[\mathcal{O}],[\mathcal{I}]\}$

P r o o f. The first point is to observe that the harmonic decomposition of $\mathbb{G}^{2n+1} := \otimes^{2n+1}\mathbb{R}^3$ contains harmonic spaces ranging from order 0 to $2n+1$ with multiplicity greater or equal to one. Using the Clebsch-Gordan rule [2,21], the tensor product of $\mathbb{H}^i$ and $\mathbb{H}^j$ can be expanded in the following way:

$$\mathbb{H}^i \otimes \mathbb{H}^j = \bigoplus_{k=|i-j|}^{i+j} \mathbb{H}^k$$

Therefore we have

$$\begin{cases} k=0, \ \mathbb{H}^0 \otimes \mathbb{H}^1 = \mathbb{H}^1 \\ k \neq 0, \ \mathbb{H}^k \otimes \mathbb{H}^1 = \mathbb{H}^{k+1} \oplus \mathbb{H}^k \oplus \mathbb{H}^{k-1} \end{cases}$$

Because $\mathbb{G}^{2n+1} = \mathbb{G}^{2n} \otimes \mathbb{H}^1$, itering this rule demonstrates the property of the "completeness" of the decomposition, it remains to apply the theorem 5.1 to conclude the proof. $\square$

Now, the planar symmetry classes of $\mathbb{G}^{2n+1}_\star$ and $\mathbb{G}^{2n+1}$ have to be characterized. Let us begin with $\mathbb{G}^{2n+1}_\star$ which was defined to be the space $\mathbb{G}^{2n+1}$ without scalars,

$$\mathbb{G}^{2n+1} = \mathbb{G}^{2n+1}_\star \oplus \alpha_0 \mathbb{H}^0 \quad , \quad \alpha_0 \in \mathbb{N}^\star$$

Lemma 5.5 *The planar symmetry classes of $\mathbb{G}^{2n+1}_\star$ are:*

$$\begin{aligned} n \geq 1, \ \mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1}_\star)) \ &= \ \{[\mathbb{1}],[\mathrm{Z}_2],\cdots,[\mathrm{Z}_{2n+1}],[\mathrm{D}^v_2],\cdots,[\mathrm{D}^v_{2n+1}],[\mathrm{Z}^-_2],\cdots,[\mathrm{Z}^-_{4n}], \\ &[\mathrm{D}_2],\cdots,[\mathrm{D}_{2n+1}],[\mathrm{D}^h_4],\cdots,[\mathrm{D}^h_{2(2n+1)}],[\mathrm{SO}(2)],[\mathrm{O}(2)],[\mathrm{O}(2)^-],[\mathrm{O}(3)]\} \end{aligned}$$

P r o o f. First consider the decomposition

$$\mathbb{G}^{2n+1}_\star = \mathbb{S}^{2n+1} \oplus \mathbb{C}^{2n}_\star \text{ with } \mathbb{C}^{2n}_\star = \bigoplus_{k=1}^{2n} \alpha'_k \mathbb{H}^k \quad \text{with } \alpha'_k = \begin{cases} \alpha_k - 1 \text{ for } k \text{ odd} \\ \alpha_k \text{ for } k \text{ even} \end{cases}$$

As the tensor space is generic, its decomposition contains harmonic spaces of each order (c.f. proof of lemma.5.4). Providing $n \geq 1$, $\mathbb{H}^{2n}$ is not the scalar part of the decomposition and therefore its multiplicity in $\mathbb{C}_\star$ is at least equal to 1. It will be shown that:

$$\mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1}_\star)) = \mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) \circledcirc \mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n}))$$

i.e. the symmetry classes of $\mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1}_\star))$ are generated only by the product of its symmetric part $\mathbb{S}^{2n+1}$ with the higher-order space in $\mathbb{H}^{2n}$. Using theorem (5.3), we have

$$\begin{aligned} n \geq 2, \ \mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1})) \ &= \ \{[\mathbb{1}],[\mathrm{Z}_2],\cdots,[\mathrm{Z}_{2n-1}],[\mathrm{D}^v_2],\cdots,[\mathrm{D}^v_{2n+1}],[\mathrm{Z}^-_2],\cdots,[\mathrm{Z}^-_{2(2n-1)}], \\ &[\mathrm{D}_2],\cdots,[\mathrm{D}_n],[\mathrm{D}^h_4],\cdots,[\mathrm{D}^h_{2(2n+1)}],[\mathrm{O}(2)^-],[\mathrm{O}(3)]\} \end{aligned}$$

In this list the following classes are missing:

$$\{[Z_{2n}],[Z_{2n+1}],[SO(2)],[D_{2n}],[D_{2n+1}],[O(2)],[Z^-_{4n}],[Z^-_{2(2n+1)}],[SO(3)]\}$$

The clips of $\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1}))$ with $\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n}))$ generates new classes. More precisely, using theorem (4.7) and table 1, we have the following results

$$\begin{array}{rcl} [D^v_k]\circledcirc[O(2)]=[Z_k]\circledcirc[O(2)]=[Z_k] & ; & [O(2)^-]\circledcirc[O(2)]=[SO(2)]\circledcirc[O(2)]=[SO(2)] \\ [D^h_{2k}]\circledcirc[O(2)]=[D_k]\circledcirc[O(2)]=[D_k] & ; & [O(3)]\circledcirc[O(2)]=[O(2)] \end{array}$$

And, as the collection of $[D^v_k]$ and $[D^h_{2(2k+1)}]$ are complete, these relations provide the missing type I classes. For the classe $[Z^-_{2k}]$, the product $[D^h_{4n}]\circledcirc[D^h_{4n}]$ generates $[Z^-_{4n}]$, but no clips operation can generate the remaining classes $[Z^-_{2(2n+1)}]$ and $[SO(3)]$. Furthermore neither adding lower odd-order harmonic space, nor adding another $\mathbb{H}^{2n}$ space change the situation. This concludes the proof. Therefore as indicated

$$\mathcal{P}(\mathfrak{I}(\mathbb{G}^{2n+1}_\star))=\mathcal{P}(\mathfrak{I}(\mathbb{S}^{2n+1}))\circledcirc\mathcal{P}(\mathfrak{I}(\mathbb{H}^{2n}))$$

□

Lemma 5.6 *The symmetry classes of* $\mathbb{G}^{2n+1}$ *are:*

$$\mathfrak{I}(\mathbb{G}^{2n+1})=\mathfrak{I}(\mathbb{G}^{\star(2n+1)})\cup\{[SO(3)]\}$$

Proof. As below $\mathbb{G}^{2n+1}$ can be decomposed into $\mathbb{G}^{2n+1}_\star\oplus\alpha_0\mathbb{H}^0$. As a consequence of corollary 5.8 and lemma 5.5, its symmetry classes can be expressed as

$$\mathfrak{I}(\mathbb{G}^{2n+1}) \quad = \quad \mathfrak{I}(\mathbb{G}^{2n+1}_\star)\circledcirc\mathfrak{I}(\mathbb{H}^0)$$

In the collection $\mathfrak{I}(\mathbb{G}^{2n+1}_\star)$ the following classes are missing:

$$\{[Z^-_{2(2n+1)}],[SO(3)]\}$$

As $\mathfrak{I}(\mathbb{H}^0)=\{[SO(3)],[O(3)]\}$, the clips operation $[SO(3)]\circledcirc[O(3)]$ gives $[SO(3)]$. But $[Z^-_{2(2n+1)}]$ can not be generated by any clips, and so only one new class appears. □

Therefore, the combination of this result with the characterization of the spatial one leads to:

n	1	2	≥ 3
$\#\mathfrak{I}(\mathbb{G}^{2n+1}_\star)$	16	28	$10n+9$
$\#\mathfrak{I}(\mathbb{G}^{(2n+1)})$	17	29	$10(n+1)$

5.2.3 Constitutive tensor space

Let us consider now a constitutive tensor space $\mathbb{T}^{2n+1}_c=\mathbb{E}_1\otimes\mathbb{E}_2$ together with its O(3)-irreducible decomposition:

$$\mathbb{T}^{2n+1}_c\simeq\bigoplus_{k=0}^{2n+1}\alpha_k\mathbb{H}^k$$

The following property can be stated:

Proposition 5.7 *Any constitutive tensors spaces* $\mathbb{T}^{2n+1}_c$ *between* $\mathbb{E}_1=\mathbb{T}^{2p+1}$ *and* $\mathbb{E}_2=\mathbb{T}^{2q}$*, such as* $q\neq 0$*, contains* $\mathbb{H}^k$ *for each* $1\leq k\leq 2n+1$*.*

Proof. We suppose here that $2p+1>2q$, the spaces $\mathbb{T}^{2p+1}$ and $\mathbb{T}^{2q}$ contain respectively $\mathbb{S}^{2p+1}$ and $\mathbb{S}^{2q}$ as subspaces. Now it is clear that

$$\mathbb{H}^{2q}\otimes\mathbb{S}^{2p+1}\subseteq\mathbb{S}^{2q}\otimes\mathbb{S}^{2p+1}\subseteq\mathbb{T}^{2(p+q)+1}=\mathbb{T}^{2n+1}_c \text{ with } \mathbb{S}^{2p+1}=\bigoplus_{k=0}^{p}\mathbb{H}^{2k+1}$$

Thus the space $\mathbb{T}^{2n+1}_c$ contains $\mathbb{H}^{2q}\otimes\mathbb{H}^{2k+1}$ for $0\leq k\leq p$.
By application of the Clebsch-Gordan rule, each tensor product $\mathbb{H}^{2q}\otimes\mathbb{H}^{2k+1}$ contains spaces $\mathbb{H}^i$ for i from $|2q-2k-1|$ to $2q+2k+1$ and for k from 0 to p, which give us the spaces $\mathbb{H}^1$, $\mathbb{H}^2$, $\cdots$, $\mathbb{H}^{2p+2q+1}$. The same kind of proof can be given in the case when $2q>2p+1$. □

A first consequence of this proposition is that $\mathbb{S}^{2n+1} \subset \mathbb{T}_c^{2n+1} \subseteq \mathbb{G}^{2n+1}$. Therefore the symmetry classes of $\mathbb{T}_c^{2n+1}$ always differ from the ones of $\mathbb{S}^{2n+1}$. Another consequence of proposition.5.7 is

Corollary 5.8 *Let $\mathbb{T}_c^{2n+1}$ be a constitutive tensor space which $\mathrm{O}(3)$-irreducible decomposition is $\mathbb{T}_c^{2n+1} \simeq \bigoplus_{k=0}^{2n+1} \alpha_k \mathbb{H}^k$, we have $\alpha_{2n} \geq 1$.*

Proposition 5.9 *A constitutive tensors space $\mathbb{T}_c^{2n+1}$ between $\mathbb{E}_1 = \mathbb{T}^{2p+1}$ and $\mathbb{E}_2 = \mathbb{T}^{2q}$, $q \neq 0$, contains $\mathbb{H}^0$ if and only if the harmonic decompositions of $\mathbb{T}^{2p+1}$ and $\mathbb{T}^{2q}$ contain, at least, one $\mathbb{H}^k, k \geq 0$ in common.*

Proof. By the mean of the Clebsch-Gordan product, the only way to obtain a space $\mathbb{H}^0$, is to combine same order harmonic spaces in the product of $\mathbb{E}_1$ by $\mathbb{E}_2$. Therefore, once there is, at least, one term in common in the harmonic decomposition of $\mathbb{T}^{2q}$ and $\mathbb{T}^{2p+1}$, the space $\mathbb{T}_c^{2n+1}$ contains a scalar component. □

And we can conclude with the following theorem

Theorem 5.10 *Let consider $\mathbb{T}_c^{2n+1}$ the space of coupling tensors between two physics described respectively by two tensor spaces $\mathbb{E}_1$ and $\mathbb{E}_2$. If these tensor spaces are of orders greater or equal to 1, then*

- $\mathfrak{I}(\mathbb{T}_c^{2n+1}) = \mathfrak{I}(\mathbb{G}_\star^{2n+1})$ *if $\mathbb{E}_1 = \mathbb{I}^{2p+1}$ (resp $\mathbb{E}_2 = \mathbb{I}^{2p+1}$) is a space of odd-order tensors which harmonic decomposition only contains odd-order terms, and $\mathbb{E}_2 = \mathbb{P}^{2q}$ (resp $\mathbb{E}_1 = \mathbb{P}^{2q}$) a space of even-order tensors which harmonic decomposition only contains even-order terms;*
- $\mathfrak{I}(\mathbb{T}_c^{2n+1}) = \mathfrak{I}(\mathbb{G}^{2n+1})$ *otherwise.*

Proof. By a consequence of proposition 5.7 any odd-order constitutive tensor space $\mathbb{T}_c^{2n+1}$ contains $\mathbb{H}^{2n-1}, \mathbb{H}^{2n}$ and $\mathbb{H}^{2n+1}$ in its harmonic decomposition. And, as proved by lemma 5.5, the clips operations between these spaces generate $\mathcal{P}(\mathfrak{I}(\mathbb{G}_\star^{2n+1}))$. For the spatial class proposition 5.7 allows to use the lemma 5.4. Therefore, at least, $\mathfrak{I}(\mathbb{T}_c^{(2n+1)}) = \mathfrak{I}(\mathbb{G}_\star^{(2n+1)})$. If furthermore the harmonic decomposition of $\mathbb{E}_1$ and $\mathbb{E}_2$ contain, at least, two harmonic spaces of the same order, the class $[\mathrm{SO}(3)]$ is added to the isotropy classes, and $\mathfrak{I}(\mathbb{T}_c^{(2n+1)}) = \mathfrak{I}(\mathbb{G}^{(2n+1)})$, which concludes the proof. □

6 Conclusion

Extending results concerning the symmetry class determination for even-order tensor spaces [28], we obtain in this paper a general theorem that gives the number and type of symmetry classes for any odd-order tensor spaces. Therefore the combination of those two results completely solves the question. Application of our results is direct and solves directly problems that would have been difficult to manage with the Forte-Vianello method. As an example, and for the first time, the symmetry classes of the odd-order tensors involved in Mindlin second strain-gradient elasticity were given. To reach this goal a geometric tool, called the clips operator, has been introduced. Its main properties and all the products for SO(3) andO(3)-closed-subgroups were also provided. We believe that these results may find applications in other contexts.

A Clips operation on $\mathrm{O}(3)$-subgroups

Here we establish results concerning the clips operator on O(3)-subgroups. The geometric idea to study the intersection of symmetry classes relies on the symmetry determination of composite figures which are the intersection of two elementary figures. As an example we consider the rotation $\mathbf{r} = \mathbf{Q}(\mathbf{k}; \frac{\pi}{3})$; determining $\mathrm{D}_4 \cap \mathbf{r}\mathrm{D}_4\mathbf{r}^t$ is tantamount to establishing the set of transformations letting the composite Figure 7 invariant.

A.1 Axes and plane reflection

In this subsection the structure of the closed O(3)-subgroups will be defined. Their abstract determination can be found in [20, 30] and will not be detailed here. Each closed O(3)-subgroup can either contain $-\mathbb{1}$ or not. Let Z_2^c denotes the center of O(3). It worth noting that despite being isomorphic, Z_2^c and Z_2 are not conjugate. Let Γ be a closed O(3)-subgroups, we have the following alternative:

- $-\mathbb{1} \in \Gamma$, therefore there exists and isomorphism between Γ and a group $K \oplus \mathrm{Z}_2^c$ where K is a closed SO(3)-subgroup. Those groups are said to be of type II.

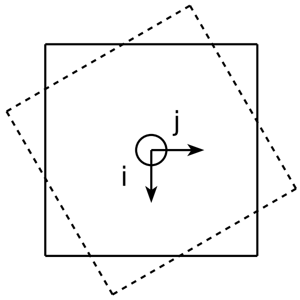

Fig. 7 Composite figure associated to $\mathrm{D}_4 \cap \mathbf{r}\mathrm{D}_4\mathbf{r}^t$ where $\mathbf{r} = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{3}\right)$

- $-\mathbb{1} \notin \Gamma$, in such case an other alternative appears
  - Γ is a closed SO(3)-subgroup, those groups are said to be of type I;
  - Γ is not a closed SO(3)-subgroup; in such case two SO(3)-subgroups can be found such as $H \subset L$ and L has indice two in H. These two SO(3)-subgroups entirely determine Γ. Those groups are said to be of type III.

The pairs of SO(3)-subgroups generating type III subgroups are (see [20]):

$$\mathrm{SO}(2) \subset \mathrm{O}(2),\ \mathcal{T} \subset \mathcal{O},\ \mathrm{Z}_n \subset \mathrm{D}_n,\ \mathrm{D}_n \subset \mathrm{D}_{2n},\ \mathrm{Z}_n \subset \mathrm{Z}_{2n} \text{ and } \mathbb{1} \subset \mathrm{Z}_2$$

Construction of each subgroup related to these couples will be provided in the following of this subsection. The main idea [20] is that we will have $\Gamma = H \cup (-\gamma H)$ where $\gamma \in \Gamma - H$.

We now define the transformation σ_z to be the reflection through the plane normal to Oz axis. Similarly, we define σ_b to be the reflection through the plan normal to b axis. Now:

- The group Z_2^- is generated by σ_z. This group is isomorphic to Z_2 and Z_2^c but non conjugate to them. Conjugate groups to Z_2^- will be noted $\mathrm{Z}_2^{\sigma_b}$ and defined as

  $$\mathrm{Z}_2^{\sigma_b} := \{\mathbb{1}, \sigma_b\}$$

- For the two subgroups $\mathrm{Z}_{2n} \supset \mathrm{Z}_n$ (where $n > 1$), Z_{2n} is the group of rotations

  $$\mathbb{1}, \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right), \mathbf{Q}\left(\mathbf{k}; \frac{2\pi}{n}\right), \cdots$$

  We can take here $\gamma = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right) \in \mathrm{Z}_{2n} - \mathrm{Z}_n$ and we will obtain

  $$\mathrm{Z}_{2n}^- := \mathrm{Z}_n \cup \mathrm{Z}_n^\nu \text{ where } \mathrm{Z}_n^\nu := -\gamma \mathrm{Z}_n$$

  It can be observed that each element of $-\gamma\mathrm{Z}_n$ is a rotation multiplied by a reflection in the xy plane. Z_{2n}^- has a primary axis[20] Oz, and for another axis a, the associated conjugate subgroup will be noted $\mathrm{Z}_{2n}^-(a)$

- Let consider $\mathrm{D}_n \supset \mathrm{Z}_n$, D_n is composed with all the Z_n rotations together with two orders rotations around axis denoted b_j and orthogonal to the primary axis Oz of Z_n. Each of these two-order rotation can be multiplied with $-\mathbb{1}$, leading to

  $$-\mathbf{Q}(b; \pi) = \sigma_b$$

  which is a plane reflection as defined above. The dihedral-v subgroup D_n^v is obtained[21]

  $$\mathrm{D}_n^v := \mathrm{Z}_n \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{\sigma_{b_j}} \tag{5}$$

  The subscript v denote the fact that we have reflection with vertical planes. If now we take the primary axis of D_n to be a and a secondary axis to be b, we denote $\mathrm{D}_n^v(a, b)$ to be the dihedral-v subgroup associated.

[20] As in [28] we define the primary axis of a cyclic group or a dihedral group to be its rotation axis ; a secondary axis of a dihedral group is related to each axial symmetry

[21] We use here the same notation as in [28]: $\biguplus \Gamma_k$ meaning the union of groups Γ_k having only identity in common

- For the two subgroups $\mathrm{D}_{2n} \supset \mathrm{D}_n$ let's define the following sets of axis p_j and q_j ($j = 0 \cdots n-1$) generated, respectively, by

$$\mathbf{Q}\left(\mathbf{k};\frac{2j\pi}{n}\right)\cdot\mathbf{i} \quad \text{for } p_j; \quad q_j := \mathbf{Q}\left(\mathbf{k};\frac{(2j+1)\pi}{2n}\right)\cdot\mathbf{i} \text{ for } q_j$$

  The dihedrale group D_n can be extracted from D_{2n}, using again the element $\gamma = \mathbf{Q}\left(\mathbf{k};\frac{\pi}{n}\right)$:

$$\mathrm{D}_n = \left\{\mathbb{1}, \mathbf{Q}\left(\mathbf{k};\frac{2\pi}{n}\right), \mathbf{Q}\left(\mathbf{k};\frac{4\pi}{n}\right), \cdots, \mathbf{Q}\left(p_0;\pi\right), \mathbf{Q}\left(p_1;\pi\right), \cdots\right\}$$

  and the following set constructed

$$\mathrm{D}_n^{\nu} := -\gamma\mathrm{D}_n = \left\{-\mathbf{Q}\left(\mathbf{k};\frac{\pi}{n}\right), -\mathbf{Q}\left(\mathbf{k};\frac{3\pi}{n}\right), \cdots, -\mathbf{Q}\left(q_0;\pi\right), -\mathbf{Q}\left(q_1;\pi\right), \cdots\right\}$$

  which elements are not in D_n. The dihedral-h subgroup is then defined

$$\mathrm{D}_{2n}^{h} := \mathrm{D}_n \cup \mathrm{D}_n^{\nu}$$

  which can be decomposed into

$$\mathrm{D}_{2n}^{h} = \mathrm{Z}_{2n}^{-} \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{p_j} \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{\sigma_{q_j}} \tag{6}$$

  where we have n copies of Z_2 and n copies of Z_2^{-}. The subscript h denotes the fact that we have reflections with horizontal planes. As for the dihedral-v subgroups, for each primary axis a and for a secondary b axis, we will denote $\mathrm{D}_{2n}^{h}(a,b)$ the dihedral-h subgroup associated.

- For the two subgroups $\mathcal{O} \supset \mathcal{T}$, the following decomposition is used [22]

$$\mathcal{O} = \biguplus_{i=1}^{3} \mathrm{Z}_4^{f_i} \biguplus_{j=1}^{4} \mathrm{Z}_3^{v_j} \biguplus_{l=1}^{6} \mathrm{Z}_2^{e_l} = \mathcal{T} \cup (\mathcal{O} - \mathcal{T})$$

  with

$$\mathcal{T} = \biguplus_{j=1}^{4} \mathrm{Z}_3^{v_j} \uplus \mathrm{Z}_2^{et_1} \uplus \mathrm{Z}_2^{et_2} \uplus \mathrm{Z}_2^{et_3}$$

  where in fact $\mathrm{Z}_2^{et_i} \subset \mathrm{Z}_4^{f_i}$. Then, multiplying each element of $\mathcal{O} - \mathcal{T}$ by $-\mathbb{1}$ the subgroup $\mathcal{O}^{-}$is obtained as:

$$\mathcal{O}^{-} := \biguplus_{i=1}^{3} \mathrm{Z}_4^{-}(f_i) \biguplus_{j=1}^{4} \mathrm{Z}_3^{v_j} \biguplus_{l=1}^{6} \mathrm{Z}_2^{\sigma_{e_l}} \tag{7}$$

  in which we can see six plane reflections, and those planes are perpendicular to edge axes.

- For the last case $\mathrm{O}(2) \supset \mathrm{SO}(2)$ we observe that each two-order rotation $\mathbf{Q}(\mathbf{x};\pi)$, with $\mathbf{x}$ in the horizontal plane xy is not in $\mathrm{SO}(2)$, and thus we can multiply each of these rotations by $-\mathbb{1}$, we then obtain a plane reflection $\sigma_{\mathbf{x}}$ and we have

$$\mathrm{O}(2)^{-} := \mathrm{SO}(2) \bigcup_{\mathbf{x}\in xy} \mathrm{Z}_2^{\sigma_{\mathbf{x}}}$$

We give now the partially ordered set[23] of conjugacy classes related to $[\mathcal{O}^{-}]$ in figure 8 and of conjugacy classes related to other closed subgroups of $\mathrm{O}(3)$ in figure 9.

[22] we have use simplified notations to denote axes of the cube's subgroups: $f_i := fc_i$, $e_l = ec_l$ and $v_j := vc_j$
[23] An arrow between $[\Sigma_1]$ and $[\Sigma_2]$ means there exist $g \in \mathrm{O}(3)$ such that $g\Sigma_1 g^{-1} \subset \Sigma_2$

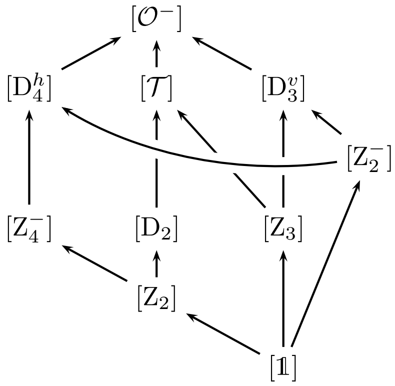

Fig. 8 Partially ordered set of conjugacy classes of closed subgroups of $\mathcal{O}^-$

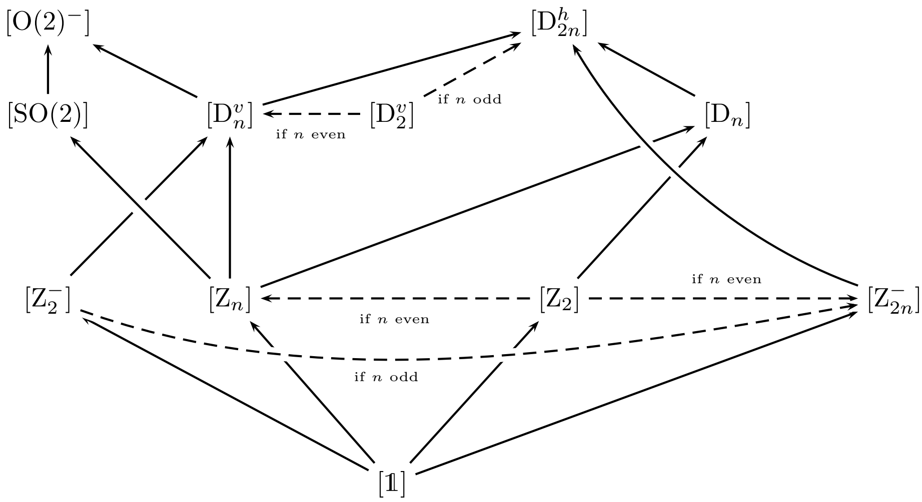

Fig. 9 Partially ordered set of conjugacy classes of closed subgroups of O(3)

A.2 Clips operations

It should be noted that $-\mathbb{1} \in \mathrm{O}(3)$ can act in the given vector space $\mathcal{E}$ either as $-id_{\mathcal{E}}$ or as $id_{\mathcal{E}}$. In the second case, the representation correspond with a SO(3)-representation, meanwhile in the first, for every $\mathbf{x} \in \mathcal{E}$,

$$(-\mathbb{1}) \cdot \mathbf{x} = -\mathbf{x}$$

Therefore in this case $-\mathbb{1}$ can never be in an isotropy subgroup, therefore excluding type II subgroups. Thus, attention will be focused on type I and III subgroups. We have the obvious lemma

Lemma A.1 *For every subgroup Σ_1^- of class III and for every subgroup Σ_2 of class I, we have*

$$\Sigma_1^- \cap \Sigma_2 = (\Sigma_1^- \cap \mathrm{SO}(3)) \cap \Sigma_2$$

This lemma allows to deduce all clips operations between class I and class III subgroups using proofs of [28]. Clips operations for class III subgroups have to be studied case by case. The following notation will be adopted

$$\mathrm{Z}_1^\sigma := \mathbb{1}\ ;\ \mathrm{Z}_1^- := \mathbb{1};\ \mathrm{D}_1^v := \mathbb{1}$$

Now, we will only give some sketches of the proofs: the main ideas had been detailled in our previous paper [28]. Nevertheless, we recall here that all arguing are based on geometric relations between axes of subgroups. Let's take for exemple the determination of the clips operation

$$\left[\mathcal{O}^-\right] \circledcirc \left[\mathrm{Z}_2^-\right]$$

All we have to observe is the axis relation between Z_2^- and the decomposition of $\mathcal{O}^-$ given by 7 : either we have a common axis with $\mathrm{Z}_2^{\sigma_{e_l}}$ or not ; which gives the wanted result. Now the following lemma is easily obtained:

Lemma A.2 *For every integer $n \geq 2$ we have*

$$[\mathrm{D}_n^v] \circledcirc [\mathrm{Z}_2^-] = [\mathrm{D}_{2n}^h] \circledcirc [\mathrm{Z}_2^-] = [\mathcal{O}^-] \circledcirc [\mathrm{Z}_2^-] = [\mathrm{O}(2)^-] \circledcirc [\mathrm{Z}_2^-] = \{\mathbb{1}, [\mathrm{Z}_2^-]\}$$

And, if we define

$$i(n) := 3 - gcd(2,n) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 2 \text{ if } n \text{ is odd} \end{cases}$$

then

$$[\mathrm{Z}_{2n}^-] \circledcirc [\mathrm{Z}_2^-] = \left\{\mathbb{1}, \left[\mathrm{Z}_{i(n)}^-\right]\right\}$$

Proof. The first two clips operations are obvious. For the last one, we have to observe that $\sigma = -\mathbf{Q}(\mathbf{k};\pi) \in \mathrm{Z}_{2n}^-$ only in the case when n is odd. □

Similarly, arguing on axes and on decompositions of subgroups, the following lemma can be established:

Lemma A.3 *For every integer n, we note $i(n)$ as in lemma A.2. We have:*

$$[\mathrm{O}(2)^-] \circledcirc [\mathrm{Z}_{2n}^-] = \{[\mathbb{1}], [\mathrm{Z}_{i(n)}^-], [\mathrm{Z}_n]\} \quad ; \quad [\mathrm{O}(2)^-] \circledcirc [\mathrm{D}_n^v] = \{[\mathbb{1}], [\mathrm{Z}_2^-], [\mathrm{D}_n^v]\}$$

$$[\mathrm{O}(2)^-] \circledcirc [\mathrm{D}_{2n}^h] = \{[\mathbb{1}], [\mathrm{Z}_{i(n)}], [\mathrm{Z}_2^-], [\mathrm{D}_{i(n)}^v], [\mathrm{D}_n^v]\} \quad ; \quad [\mathrm{O}(2)^-] \circledcirc [\mathcal{O}^-] = \{[\mathbb{1}], [\mathrm{Z}_2^-], [\mathrm{D}_3^v], [\mathrm{D}_2^v]\}$$

$$[\mathrm{O}(2)^-] \circledcirc [\mathrm{O}(2)^-] = \{[\mathrm{Z}_2^-], [\mathrm{O}(2)^-]\}$$

Z_{2n}^- situations

The next lemma directly leads us to the sought result.

Lemma A.4 *For every integers m and n, we note $d = gcd(n,m)$. Then*

1. *Either $d = 1$ and n and m are odds then $\mathrm{Z}_n^\nu \cap \mathrm{Z}_m^\nu = \{-\mathbf{Q}(\mathbf{k};\pi)\}$;*
2. *Either $d \neq 1$ and $\dfrac{n}{d}$ and $\dfrac{m}{d}$ are odds, then $\mathrm{Z}_n^\nu \cap \mathrm{Z}_m^\nu = \mathrm{Z}_d^\nu$*
3. *Otherwise $\mathrm{Z}_n^\nu \cap \mathrm{Z}_m^\nu = \emptyset$*

Proof. Taking m_1 and n_1 to be such that $m = dm_1$ and $n = dn_1$, it can be observed that the intersection is not empty iff there exists j and l such as

$$(2j+1)\frac{\pi}{dn_1} = (2l+1)\frac{\pi}{dm_1} \text{ and then } (2j+1)m_1 = (2l+1)n_1$$

and this equality shows that neither n_1 nor m_1 can be even. □

The following lemma is a consequence of lemma A.4

Lemma A.5 *For every integers n and m, we note $d = gcd(n,m)$ and*

$$i1(m,n) := \begin{cases} 2d \text{ if } \dfrac{n}{d} \text{ and } \dfrac{m}{d} \text{ are odd} \\ 1 \text{ otherwise} \end{cases} \quad ; \; i2(m,n) := \begin{cases} 1 \text{ if } \dfrac{n}{d} \text{ and } \dfrac{m}{d} \text{ are odd} \\ d \text{ otherwise} \end{cases}$$

Then

$$[\mathrm{Z}_{2n}^-] \circledcirc [\mathrm{Z}_{2m}^-] = \left\{\mathbb{1}, \left[\mathrm{Z}_{i1(m,n)}^-\right], [\mathrm{Z}_{i2(m,n)}]\right\}$$

In fact, the idea is to compute the gcd of m and n then to study the parity of the two integers $\frac{n}{d}$ and $\frac{m}{d}$: when the two are odds we know that $\mathrm{Z}_{2d}^- \subset \mathrm{Z}_{2n}^-$ and $\mathrm{Z}_{2d}^- \subset \mathrm{Z}_{2m}^-$. Thus when these two subgroups have the same primary axis, we will have $\mathrm{Z}_{2n}^- \cap \mathrm{Z}_{2m}^- = \mathrm{Z}_{2d}^-$; then the clips operation is $\{\mathbb{1}, [\mathrm{Z}_{2d}^-]\}$. Furthermore we can have Z_2^- when $d = 1$ and the two integers m and n are odds. In the other case we only have $\{\mathbb{1}, [\mathrm{Z}_d]\}$.

D_n^v situations

Let consider the decomposition (5):

$$\mathrm{D}_n^v := \mathrm{Z}_n^0 \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{\sigma_{b_j}}$$

And we recall that Z_n^0 has Oz as primary axis, and we let b_0, a secondary axis, generated by $\mathbf{i}$. For each rotation $g \in \mathrm{SO}(3)$, a is taken to be the axis generated by $g\mathbf{k}$, and therefore one has to study

$$\Gamma = \mathrm{D}_n^v \cap \mathrm{Z}_{2m}^-(a)$$

From the construction of $\mathrm{Z}_{2m}^-(a)$ we have:

$$\mathrm{Z}_{2m}^-(a) = \mathrm{Z}_m^a \cup (\mathrm{Z}_m^a)^\nu$$

There is only two cases:

- either $a \neq Oz$: the only way to have something else than identity is to have, in $(\mathrm{Z}_m^a)^\nu$ some reflections $-\mathbf{Q}(\mathbf{x};\pi)$ where $\mathbf{x}$ generates some of the b_j. This situation happens if $g\mathbf{k} = \mathbf{x}$ and if m is odd (to have a $\mathbf{Q}(\mathbf{k};\pi)$ rotation in Z_{2m}). Thus, as soon as m is odd Z_2^- is conjugate to Γ.
- either $a = Oz$: Γ reduces to $\mathrm{Z}_n^0 \cap \mathrm{Z}_m^0$ and we directly get Z_d^0 where $d = gcd(m, n)$.

We have then prove the lemma:

Lemma A.6 *For each integer $n \geq 2$ and for each integer $m \geq 2$, we note*

$$i(m) := 3 - gcd(2, m) = \begin{cases} 1 \text{ if } m \text{ is even} \\ 2 \text{ if } m \text{ is odd} \end{cases} \qquad \text{and } d = gcd(n, m)$$

Then

$$[\mathrm{D}_n^v] \circledcirc [\mathrm{Z}_{2m}^-] = \left\{\mathbb{1}, \left[\mathrm{Z}_{i(m)}^-\right], [\mathrm{Z}_d]\right\}$$

Arguing on primary and secondary axis leads to the next lemma:

Lemma A.7 *For each integers $n \geq 2$ and $m \geq 2$, we note $d = gcd(n, m)$. We then have*

$$[\mathrm{D}_n^v] \circledcirc [\mathrm{D}_m^v] = \left\{\mathbb{1}, [\mathrm{Z}_2^-], [\mathrm{D}_d^v], [\mathrm{Z}_d]\right\}$$

D_{2n}^h situation

Great use of the following obvious lemma will be made. This lemma can be proved by direct computation:

Lemma A.8 *For every integer n we note*

$$\mathrm{D}_{2n}^h = \mathrm{Z}_{2n}^- \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{p_j} \biguplus_{j=0}^{n-1} \mathrm{Z}_2^{\sigma_{q_j}}$$

with

$$q_j = \mathbf{Q}\left(\mathbf{k}; \frac{(2j+1)\pi}{2n}\right),\ p_j = \mathbf{Q}\left(\mathbf{k}; \frac{j\pi}{n}\right),\ j = 0 \cdots (n-1)$$

Then, if n is even there exists two perpendicular axes p_k, p_l and two perpendicular axes q_r, q_s, furthermore no axes p_i, q_j are perpendicular. If n is odd there exists two perpendicular axes p_i, q_j and no axes p_k, p_l nor q_r, q_s are perpendicular.

Now, arguing on axes and using lemma A.4 leads us to:

Lemma A.9 *For every integers $n \geq 2$ and $m \geq 2$, we note $d_2(m) = gcd(m, 2)$,*

$$i(m) = \begin{cases} 1 \text{ if } m \text{ is even} \\ 2 \text{ otherwise} \end{cases}$$

and $d = gcd(n, m)$. Then

- *If* $\dfrac{n}{d}$ *or* $\dfrac{m}{d}$ *is even*

$$[\mathrm{D}_{2n}^{h}] \circledcirc [\mathrm{Z}_{2m}^{-}] = \Big\{\mathbb{1}, [\mathrm{Z}_{d_2(m)}], [\mathrm{Z}_{i(m)}^{-}], [\mathrm{Z}_{d}]\Big\}$$

- *If* $\dfrac{n}{d}$ *and* $\dfrac{m}{d}$ *are odd*

$$[\mathrm{D}_{2n}^{h}] \circledcirc [\mathrm{Z}_{2m}^{-}] = \Big\{\mathbb{1}, [\mathrm{Z}_{d_2(m)}], [\mathrm{Z}_{i(m)}^{-}], [\mathrm{Z}_{2d}^{-}]\Big\}$$

Other arguments on axes, parity and uses of lemma A.8 leads us to:

Lemma A.10 *For every integers $n \geq 2$ and $m \geq 2$, we note*

$$i(m,n) := \begin{cases} 2 \text{ if } m \text{ is even and } n \text{ is odd} \\ 1 \text{ otherwise} \end{cases} \quad ; \; d_2(m) := gcd(m,2)$$

Then we have

$$[\mathrm{D}_{2n}^{h}] \circledcirc [\mathrm{D}_{m}^{v}] = \Big\{[\mathbb{1}], \Big[\mathrm{Z}_{i(m)}^{\sigma}\Big], \big[\mathrm{Z}_{d_2(m)}\big], \Big[\mathrm{D}_{i(m,n)}^{v}\Big], [\mathrm{Z}_{d}], [\mathrm{D}_{d}^{v}]\Big\}$$

After that, long argumentations on axes, parity and uses of lemma A.8 gives us the lemma:

Lemma A.11 *For every two integers m and n we note $d = gcd(n,m)$ and*

$$\Delta = [\mathrm{D}_{2n}^{h}] \circledcirc [\mathrm{D}_{2m}^{h}]$$

Then:

- *For every d:*
  - ◇ *If m and n are even, then $\Delta \supset \{[\mathrm{Z}_2], [\mathrm{D}_2]\}$;*
  - ◇ *If m and n are odds, then $\Delta \supset \{[\mathrm{Z}_2^{-}]\}$;*
  - ◇ *Otherwise $\Delta \supset \{[\mathrm{Z}_2], [\mathrm{D}_2^{v}]\}$;*
- *If $d = 1$ then*
  - ◇ *If m and n are odds, then $\Delta \supset \{[\mathrm{D}_2^{v}]\}$;*
  - ◇ *Otherwise m or n is even and $\Delta \supset \{[\mathrm{Z}_2], [\mathrm{Z}_2^{-}]\}$;*
- *If $d \neq 1$ then*
  - ◇ *If $\dfrac{m}{d}$ and $\dfrac{n}{d}$ are odds, then $\Delta \supset \{[\mathrm{Z}_{2d}^{-}], [\mathrm{D}_{2d}^{h}]\}$;*
  - ◇ *Otherwise $\dfrac{m}{d}$ or $\dfrac{n}{d}$ is even and $\Delta \supset \{[\mathrm{Z}_d], [\mathrm{D}_d], [\mathrm{D}_d^{v}]\}$;*

This lemma is then graphically given by figure 4.

$\mathcal{O}^{-}$ situation

Here we first have the two following lemma, obtain by arguing on axes:

Lemma A.12 *For every integer n, we note $d_3(n) = gcd(3,n)$. Then*

- *If n is odd*

$$[\mathcal{O}^{-}] \circledcirc [\mathrm{Z}_{2n}^{-}] = \{[\mathbb{1}], [\mathrm{Z}_2^{-}], [\mathrm{Z}_{d_3(n)}]\}$$

- *If $n = 2 + 4k$ for $k \in \mathbb{N}$*

$$[\mathcal{O}^{-}] \circledcirc [\mathrm{Z}_{2n}^{-}] = \{[\mathbb{1}], [\mathrm{Z}_4^{-}], [\mathrm{Z}_{d_3(n)}]\}$$

- *If n is even and $4 \nmid n$*

$$[\mathcal{O}^-] \circledcirc [\mathrm{Z}_{2n}^-] = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{Z}_{d_3(n)}]\}$$

This lemma is graphically given in figure 5.

Lemma A.13 *For every integer n, we note $d_2(n) = gcd(n,2)$ and $d_3(n) = gcd(n,3)$. Then we have*

$$[\mathcal{O}^-] \circledcirc [\mathrm{D}_n^v] = \{[\mathbb{1}], [\mathrm{Z}_2^-], [\mathrm{Z}_{d_3(n)}], [\mathrm{D}_{d_3(n)}^v], [\mathrm{Z}_{d_2(n)}], [\mathrm{Z}_{d_2(n)}^v]\}$$

Then, in the case of cyclic-h subgroups, arguing with lemma A.4 and arguing on axes leads to:

Lemma A.14 *For every integer n we note $d_3(n) := gcd(n,3)$*

- *If n is even and $n = 2 + 4k$ for $k \in \mathbb{N}$*

$$[\mathcal{O}^-] \circledcirc [\mathrm{D}_{2n}^-] = \{[\mathbb{1}], [\mathrm{Z}_4^-], [\mathrm{D}_4^h], [\mathrm{Z}_{d_3(n)}], [\mathrm{D}_{d_3(n)}^v]\}$$

- *If n is even and $4 \mid n$ then*

$$[\mathcal{O}^-] \circledcirc [\mathrm{D}_{2n}^-] = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{D}_2^v], [\mathrm{Z}_{d_3(n)}], [\mathrm{D}_{d_3(n)}^v]\}$$

- *If n is odd then*

$$[\mathcal{O}^-] \circledcirc [\mathrm{D}_{2n}^-] = \{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{Z}_2^-], [\mathrm{D}_2], [\mathrm{D}_2^v], [\mathrm{Z}_{d_3(n)}], [\mathrm{D}_{d_3(n)}^v]\}$$

Finally, a direct computation on axes leads and uses of lattice of $\mathcal{O}^-$ (c.f. fig.8) leads to:

Lemma A.15 *We have*

$$[\mathcal{O}^-] \circledcirc [\mathcal{O}^-] = \{[\mathbb{1}], [\mathrm{Z}_2^-], [\mathrm{Z}_4^-], [\mathrm{Z}_3]\}$$

B Coorespondance between group notations and crystallographic systems

O(3) **type I closed-subgroups**

Hermann-Maugin	Schonflies	Group
1	C_1	$\mathbb{1}$
2	C_2	Z_2
222	D_2	D_2
3	C_3	Z_3
32	D_3	D_3
4	C_4	Z_4
422	D_4	D_4
6	C_6	Z_6
622	D_6	D_6
∞	C_∞	SO(2)
$\infty 2$	D_∞	O(2)
23	T	$\mathcal{T}$
432	O	$\mathcal{O}$
532	I	$\mathcal{I}$
$\infty\infty$		SO(3)

O(3) **type II closed-subgroups**

Hermann-Maugin	Schonflies	Group
$\bar{1}$	C_i	Z_2^c
$2/m$	C_{2h}	$Z_2 \oplus Z_2^c$
mmm	D_{2h}	$D_2 \oplus Z_2^c$
$\bar{3}$	$S_6,\ C_{3i}$	$Z_3 \oplus Z_2^c$
$\bar{3}m$	D_{3d}	$D_3 \oplus Z_2^c$
$4/m$	C_{4h}	$Z_4 \oplus Z_2^c$
$4/mmm$	D_{4h}	$D_4 \oplus Z_2^c$
$6/m$	C_{6h}	$Z_6 \oplus Z_2^c$
$6/mmm$	D_{6h}	$D_6 \oplus Z_2^c$
$m\bar{3}$	T_h	$\mathcal{T} \oplus Z_2^c$
$m\bar{3}m$	O_h	$\mathcal{O} \oplus Z_2^c$
$\bar{5}\bar{3}m$	I_h	$\mathcal{I} \oplus Z_2^c$
∞/m	$C_{\infty h}$	$SO(2) \oplus Z_2^c$
∞/mm	$D_{\infty h}$	$O(2) \oplus Z_2^c$
$\infty/m\infty/m$		$O(3)$

O(3) **type III closed-subgroups**

Hermann-Maugin	Schonflies	Group
m	C_s	Z_2^-
$2mm$	C_{2v}	D_2^v
$3m$	C_{3v}	D_3^v
$\bar{4}$	S_4	Z_4^-
$4mm$	C_{4v}	D_4^v
$\bar{4}2m$	D_{2d}	D_4^h
$\bar{6}$	C_{3h}	Z_6^-
$6mm$	C_{6v}	D_6^v
$\bar{6}2m$	D_{3h}	D_6^h
$\bar{4}3m$	T_d	$\mathcal{O}^-$
∞m	$C_{\infty v}$	$O(2)^-$

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